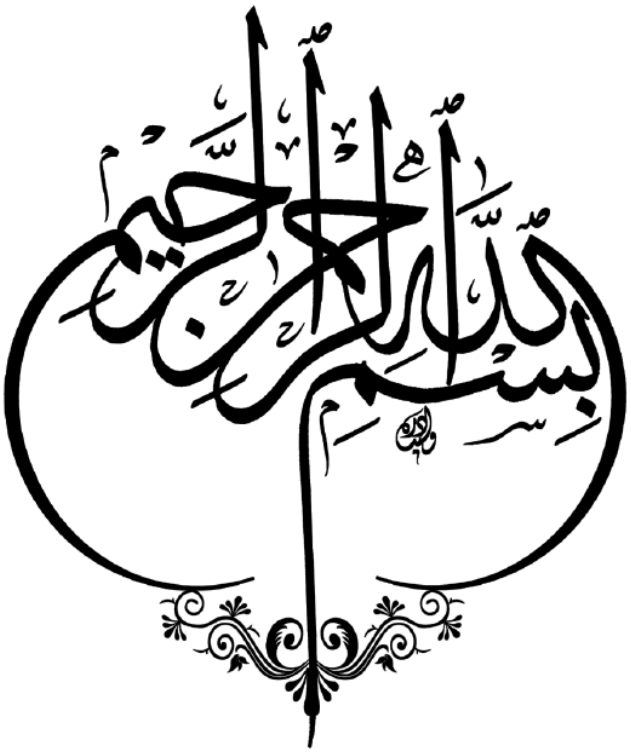
أولمبياد الموهبة الوطني
NATIONAL GIFTED OLYMPIAD

PHASE 2

Scientific Track – NSMO

Junior Mathematics

Mathematics Scientific Team



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Introduction

Our distinguished sons and daughters,

We are pleased to congratulate you on passing Phase One and qualifying for **Phase Two** — an advanced step on the path of mathematical challenge and innovation.

This package is devoted to broadening your horizons in the four main branches: **Combinatorics, Geometry, Algebra, and Number Theory**, with a focus on advanced concepts in counting, geometric review, linear equations, even and odd integers, and divisibility.

This phase aims to sharpen your analytical thinking skills and to connect mathematical concepts with one another, applying them in a variety of situations.

This package is also an opportunity to deepen your understanding of mathematical patterns and to use logic in justification and in solving problems in organized ways.

We trust in your abilities and look forward to seeing you shine in this important phase of your journey toward excellence.

The Scientific Team of the National Mawhiba Olympiad – Scientific Track
(NSMO) – Mathematics Track

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First Unit: ALGEBRA



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Linear Equations in One Variable

To solve a one-variable linear equation, follow the steps below in order:

Eliminate Denominators (If Any)

If there are fractions in the equation, multiply each term by the least common multiple (LCM) of the denominators to eliminate them.

Expand Parentheses

Use the distributive property to simplify parentheses, such as

$$2\,(x+3) = 2x + 6$$

Move Terms

Move all the terms with the variable to one side and all the constants to the other using addition and subtraction.

Combine Like Terms

Simplify both sides of the equation so that it takes the form:

$$ax = b, \text{ where } a \text{ and } b \text{ are constants.}$$

Divide by the Variable's Coefficient

We divide both sides by a (the coefficient of the variable) to get the solution:

$$ax = b \Rightarrow x = \frac{b}{a}$$

Note that:

When $a \neq 0$ we have a unique solution: $x = \frac{b}{a}$

When $a = 0$ and $b \neq 0$, there is no solution.

When $a = 0$ and $b = 0$, any real number is a solution to the equation.

Check Your Answer

We can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

Note: Sometimes we do not follow the exact order of the previous steps when a more direct method is available.

Example:

Solve the following equations:

$(a)\ \ x + 3 = 4$ $(b)\ \ x - 2 = 9$ $(c)\ \ 7x = 49$

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Solution:

$(a)\ x = 1$ $(b)\ x = 11$ $(c)\ x = 7$

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Exercises:

(1) Solve the following equations:

$(a)\ \dfrac{x}{3} = 6$

$(b)\ 2x - 1 = 19$

$(c)\ 4x - 4 = x + 11$

$(d)\ 9(x - 1) = 7(x + 1)$

(2) Solve:

$$\frac{1}{7}(5x + 2) = 1$$

(3) Solve:

$$\frac{1}{2}\left[\frac{1}{7}(5x - 1)\right] + 5 = 6$$

(4) Word Problems:

(a) If 7% of a number equals 56, what is the number?

(b) If the sum of four consecutive natural numbers is 50, what is the largest number?

(c) Two natural numbers have a ratio of 2:3, and the difference between them is 14. What is the smaller number?

(d) Ahmed bought a computer at a discount of 35% off its original price, paying 1,300 riyals. How much was the original price of the computer?

(e) Mohammed bought a car and later sold it for 46,000 riyals, making a 15% profit. What was the purchase price of the car?

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Challenge Problems (1):

(1) A train traveling at a constant speed was observed from the moment it entered a 120-meter-long tunnel until it had completely left it; the observation lasted one minute. If you know that the same train traveling at the same constant speed takes 20 seconds to pass through a traffic light in its entirety, how long is the train?

(2) We have a jar with a ratio of red balls to white balls of 1 to 4. When Saad replaces 2 white balls with 7 red balls, the ratio of red balls to white balls becomes 2 to 3. What is the ratio of the total number of balls now to the total number at the beginning?

(3) Leila gave birth to her first child on her 20th birthday, her second child exactly two years later, and her third child exactly two years after that. How old will Leila be when the combined ages of her three children equal her age?

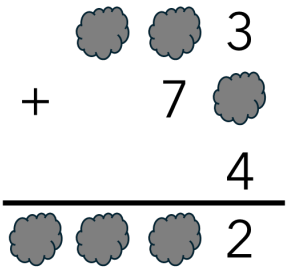
(4) A fruit basket contains apples and oranges. The ratio of apples to oranges in the basket is 3 to 8. We took one apple out of the basket. The ratio of apples to oranges in the basket became 1 to 3. How many oranges are in the basket?

(5) The arithmetic mean of six numbers is 4. When we added a seventh number, the new mean became 5. Find the seventh number.

(6) The coach was buying 17 balls for his sports club at a price of 481 riyals each. The salesman said, "You have to pay 10,177 riyals for the balls." How could he guess that he was being cheated without doing difficult calculations?

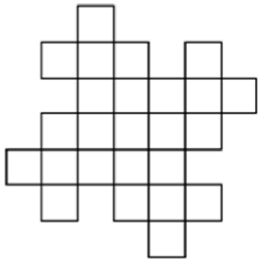
(7) The sum of two integers was multiplied by their product. Could the result be 20042401? (If so, give an example. If not, explain why.)

(8) Suad wrote an addition exercise in which all the digits are different. After proudly completing it, her little brother spilled ink on her notebook. Can you help the girl rewrite her exercise?



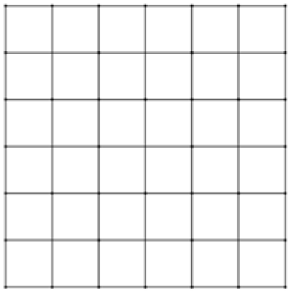
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(9) Color five squares in the figure shown so that the remaining part can be divided into five congruent pieces. (Providing one example is enough to answer the question.)



(10) A farmer bought three cows. Now each day he gets:
From the first cow: 2 liters of milk.
From the second cow: Half the amount of milk obtained from the third cow plus the same amount obtained from the first cow.
From the third cow: Half of the total amount of milk obtained from all three cows.
How many liters of milk does the farmer get in total each day?

(11) Color six squares in the grid shown black so that it becomes impossible to find any white strip of size 1×6 or any white square of size 3×3 after coloring.



(12) In the adjacent figure, the addition of three numbers is shown. Each letter represents one of the digits from 0 to 9 and stands for the same digit each time it appears. Different letters represent different digits. No number begins with the digit 0.
Find all possible values of the sum.

$$\begin{array}{r} J\ M\ O \\ J\ M\ O \\ +\ J\ M\ O \\ \hline I\ M\ O \end{array}$$

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Multi-Variable Linear Equations

> A system of linear equations consists of two separate equations that share the same two unknown variables (usually x and y).

The goal is to find the single pair of numbers (x, y) that makes both equations true at the exact same time. You can also think about the equations as two straight lines in the coordinate plane, and our goal is to find the coordinates of their intersection point.

The Elimination Algorithm

The most reliable and straightforward way to solve these systems is the **Elimination Method**. The goal is to manipulate the equations so that when you add them together, one of the variables completely disappears.

Example Problem:

Equation 1: $2x + 3y = 12$

Equation 2: $3x - y = 7$

Step 1: Line up the variables.

Ensure both equations are formatted identically. The x variables should be vertically aligned, followed by the y variables, the equals signs, and the constant numbers. (Our example is already correctly aligned).

Step 2: Force a match with opposite signs.

Look at the numbers in front of the y variables: Equation 1 has a positive 3, and Equation 2 has a negative 1. If we multiply everything in Equation 2 by 3, the y terms will perfectly cancel each other out when the equations are added together.

Multiply every term in Equation 2 by 3:

$3 \times (3x - y = 7)$ becomes $9x - 3y = 21$

Step 3: Add the equations vertically.

Stack Equation 1 and the newly adjusted Equation 2, then add straight down column by column:

$$2x + 3y = 12$$
$$9x - 3y = 21$$

The addition results in:

$$11x + 0 = 33$$

The y variable is now fully eliminated.

Step 4: Solve for the remaining variable.

You are left with a basic one-variable equation:

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$$11x = 33$$

Divide both sides by 11 to isolate x:

$$x = 3$$

Step 5: Substitute back to find the second variable.

Take your discovered value ($x = 3$) and plug it into either of the original equations to solve for y. Let's use Equation 1:

$$2\,(3) + 3y = 12$$
$$6 + 3y = 12$$

Subtract 6 from both sides:

$$3y = 6$$

Divide by 3:

$$y = 2$$

The final solution to the system is $x = 3$, $y = 2$.

Determining the Number of Solutions

Instead of memorizing abstract fraction formulas to determine how many solutions a system has, you can simply evaluate the numbers logically.

1. One Unique Solution (Intersecting Lines)

This is the standard scenario. The equations have different variable ratios and will cross at exactly one coordinate point on a graph.

Example: The problem we solved above, which yielded exactly one answer: $x = 3$, $y = 2$.

2. No Solution (Parallel Lines)

This occurs when the coefficients of the variables in both equations are perfectly proportional, but the constant terms contradict one another. The lines run parallel and never cross.

Example:

$$\begin{cases} x + y = 5 \\ x + y = 10 \end{cases}$$

It is mathematically impossible for two numbers to add up to both 5 and 10 simultaneously. Therefore, this system has zero solutions.

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3. Infinitely Many Solutions (The Same Line)

This happens when the second equation is simply a multiplied or scaled version of the first equation. Because they represent the exact same line on a graph, every point on that line is a valid solution.

Example:

$$\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$$

If you divide the entire second equation by 2, it becomes identical to the first. Any pair of numbers that adds to 5 will successfully solve both equations.

Example:

How many solutions does each of the following systems have?

(a) $\begin{cases} 5x + 2y = 4 \\ 2.5x + y = 2 \end{cases}$

(b) $\begin{cases} 4x + 3y = 3 \\ 8x + 6y = 7 \end{cases}$

Solution:

(a) Infinitely many solutions: The second equation can be obtained by dividing the first equation by 2, so they represent the same line.

(b) There is no solution: Multiplying the first by 2 gives us $8x + 6y = 6$, which contradicts the second equation (parallel lines).

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Exercises:

(1) How many solutions are there for this system?

$$\begin{cases} x + 5y = 3 \\ 2x + 10y = 6 \end{cases}$$

(2) Solve each of the following systems of equations:

(a) $\begin{cases} x + y = 13 \\ x - y = 7 \end{cases}$

(b) $\begin{cases} 2x + y = 9 \\ x + 4y = 8 \end{cases}$

(c) $\begin{cases} x + y = 14 \\ x - y = 2 \end{cases}$

(d) $\begin{cases} 2x - y = 6 \\ x + 4y = 21 \end{cases}$

(3) Two numbers have a sum of 42 and a difference of 8. What are the two numbers?

(4) At a certain time, Reema notices that her digital watch reads a minutes after two o'clock. Fifteen minutes later, it reads b minutes after three o'clock. She is amused to note that a is six times b. What time was it when she looked at her watch for the second time?

(5) Solve the following system of equations:

$$\begin{cases} \dfrac{x - y}{5} - \dfrac{x + y}{4} = \dfrac{1}{2} \\ 2(x - y) - 3(x + y) + 1 = 0 \end{cases}$$

(6) Solve the following system of equations:

$$\begin{cases} 5.4x + 4.6y = 104 \\ 4.6x + 5.4y = 96 \end{cases}$$

(7) Solve the following system of equations:

$$\begin{cases} x + 2\,(5x + y) = 16 \\ 5x + y = 7 \end{cases}$$

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(8) Solve the following system of equations:

$$\begin{cases} \frac{x}{2} = \frac{y}{3} = \frac{z}{5} \\ x + 3y + 6z = 15 \end{cases}$$

(9) Solve the following system of equations:

$$\begin{cases} x + 2y = 5 \\ y + 2z = 8 \\ z + 2u = 11 \\ u + 2x = 6 \end{cases}$$

(10) Solve the following system of equations:

$$\begin{cases} 5x - y + 3z = a \\ 5y - z + 3x = b \\ 5z - x + 3y = c \end{cases}$$

(11) Given that $x = 2, \quad y = 1$ is the solution of the system

$$\begin{cases} ax + by = 7 \\ bx + cy = 5 \end{cases}$$

what is the relation between a and c?

(12) Find the value of (a, b, c) if both systems have the same solution

$$\begin{cases} 3x - y = 5 \\ 2x + y - z = 0 \\ 4ax + 5by - z = -22 \end{cases} \quad , \quad \begin{cases} ax - by + z = 8 \\ x + y + 5 = c \\ 2x + 3y = -4 \end{cases}$$

(13) Determine the values of k for which the system of equations

$$\begin{cases} kx - y = -\frac{1}{3} \\ 3y = 1 - 6x \end{cases}$$

has: (a) a unique solution (b) no solution (c) infinitely many solutions.

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(14) Given that $x,\ y,\ z$ satisfy the system of equations

$$\begin{cases} 2020(x-y)+2021(y-z)+2022(z-x)=0 \\ 2020^2(x-y)+2021^2(y-z)+2022^2(z-x)=2021 \end{cases}$$

find the value of $z-y$.

(15) Solve the following system of equations:

$$\begin{cases} x-y-z=5 \\ y-z-x=1 \\ z-x-y=-15 \end{cases}$$

(16) Solve the following system of equations:

$$\begin{cases} x-y+z=1 \\ y-z+u=2 \\ z-u+v=3 \\ u-v+x=4 \\ v-x+y=5 \end{cases}$$

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Challenge Problems (2):

(1) Solve the following system of equations:

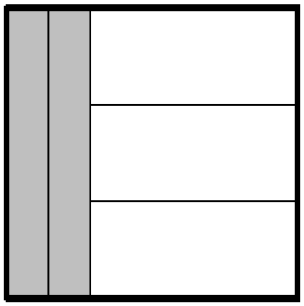
$$\begin{cases} \frac{1}{a} - \frac{1}{b} = 6 \\ \frac{1}{a} + \frac{1}{b} = 14 \end{cases}$$

(2) Solve the following system of equations:

$$\begin{cases} x + y = 8 \\ y + z = 4 \\ z + x = 6 \end{cases}$$

(3) Give an example of two fractions whose difference is equal to their product.

(4) The figure shown is a square composed of five rectangles, all having the same perimeter. What is the ratio of the area of one shaded rectangle to the area of one unshaded rectangle?



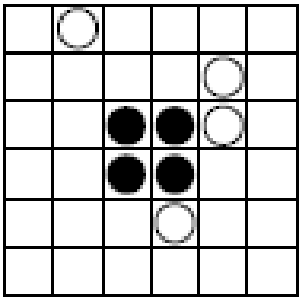
(5) We want to divide a certain amount of money equally among a group of children. If each child receives 60 halalas, 2.10 riyals will remain. However, if there were 20 halalas more than this amount, there would be just enough for each child to receive 70 halalas. How many children are in the group?

(6) The number 3600 can be written as $2^a \times 3^b \times 4^c \times 5^d$, where a, b, c, d are positive integers. If $a + b + c + d = 7$, what is the value of c?

(7) If you remove ten digits from the number $12345123451234512345123345$, what is the largest number you can get?

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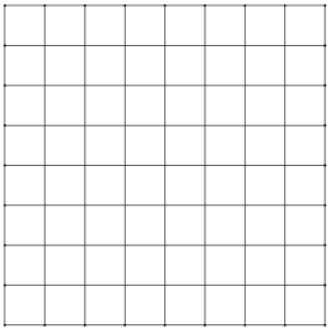
(8) Cut the following shape into four identical parts (both in shape and area) so that each part contains one black circle and one white circle.



(9) My brother has four children. They are 5, 8, 13, and 15 years old. Their names (in no particular order) are Muhammad, Raja, Najah, and Nour. One of the girls is in kindergarten, Raja is older than Muhammad, and the sum of Raja and Nour's ages is divisible by three.
Is Nour a boy or a girl?

(10) There are one hundred people living on an island. Some of them always lie, while others always tell the truth, and each of them has a favorite season of the year. Every person was asked the following four questions:
• **Do you like winter?** • **Do you like spring?** • **Do you like summer?** • **Do you like fall?**
There were 25 "yes" answers to the first question, 25 "yes" answers to the second, 45 "yes" answers to the third, and 55 "yes" answers to the fourth. How many liars are there on the island?

(11) We have the following 8×8 table divided into 64 small squares.
Can you color 17 small squares black so that no two squares share an edge or even a corner?



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Percentages

Percent literally means "per hundred." Percent is just a shorthand way of writing the ratio of a number to 100.
For example, 54% means 54 out of 100 or $\frac{54}{100}$.
Specifically, if a is x percent of b, then

$$\frac{a}{b} = \frac{x}{100}$$

For example, 3 out of 16 is written as a percentage as follows $\frac{3}{16} = \frac{x}{100}$

By simplifying, we obtain $x = 18\frac{3}{4}$

Final Answer: 3 is 18.75% of 16.

If a number, x, is increased by $k\%$, then the result is:

$$x \cdot \left(1 + \frac{k}{100}\right)$$

Similarly, if x is decreased by $k\%$, then the result is:

$$x \cdot \left(1 - \frac{k}{100}\right)$$

Example:

When a number is reduced by 40%, the result is 36. What is the original number?

Solution:

Let our number be x. We have

$$x\left(1 - \frac{40}{100}\right) = 36$$

$$\Rightarrow \frac{6}{10}x = 36$$

$$\Rightarrow x = 60$$

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Exercises:

(1) If x, y, and z are nonzero numbers that satisfy

$$\frac{7}{x+y} = \frac{k}{x+z} = \frac{11}{z-y},$$

find the value of k.

(2) Find the values of $a, b, c \in \mathbb{N}$ that satisfy the following:

$$\frac{a+1}{3} = \frac{b+2}{2} = \frac{8}{c+3}$$

(3) 36 is 120% of what number?

(4) What is the discount percentage for an item if its price changes from 2890 riyals to 2023 riyals?

(5) If Nora brought 60 cookies to school, gave 40% of them to her teachers, 25% of the rest to her friends, and ate one-third of what was left, how many cookies are left?

(6) If the ratio of the interior angles of a pentagon is $2 : 3 : 4 : 5 : 6$, what is the measure of the largest angle?

(7) If b is 5% greater than a, and b is 15% less than c, what is the ratio of a to c?

(8) A rectangle has been increased in length by 50% and in width by 20%. What is the percentage increase in its area?

(9) Every day, 20% of the fish are sold at the fish market. If 2000 fish remained at the end of Tuesday, how many fish were there at the beginning of Monday?

(10) If t is 25% of u, what is the ratio of $4t$ to $2u$?

(11) Given that $yz : zx : xy = 1 : 2 : 3$ and

$$\frac{x}{yz} : \frac{y}{zx} = 1 : k,$$

find k.

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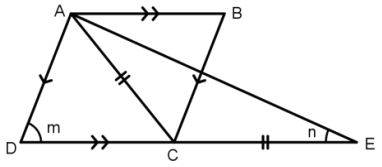
Second Unit: Geometry



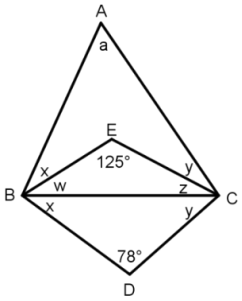
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Revision Exercises:

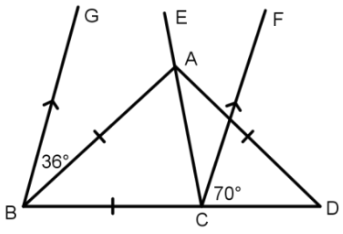
(1) In the following diagram $ABCD$ is a rhombus, ACE is an isosceles triangle with $AC = CE$ and D, E, C are on the same line. If $\angle AEC = n$ and $\angle ADC = m$, write an equation that relates m and n.



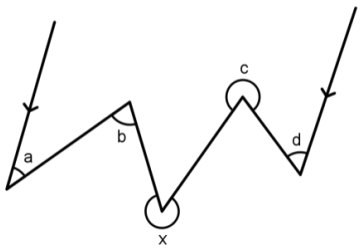
(2) In the following diagram $ABDC$ is a quadrilateral. Point E lies inside such that $\angle ABE = \angle DBC = x$, $\angle ACE = \angle DCB = y$. If $\angle BEC = 125°, \angle BDC = 78°$, find $\angle A$.



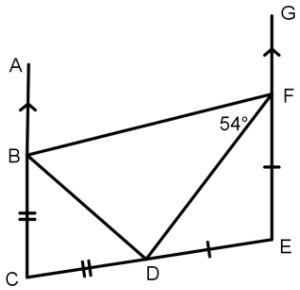
(3) In the following diagram:
Given $AB = BC = AD$, find $\angle EAD$.



(4) Find the value of x in terms of a, b, c, d.

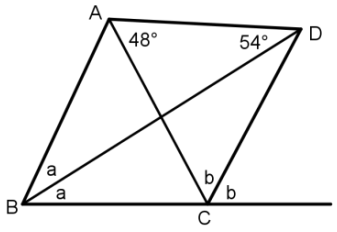


(5) In the following diagram we have $BC = CD$, $DE = EF$, $AC \parallel GE$.
Given $\angle BFD = 54°$, find $\angle DBF$.

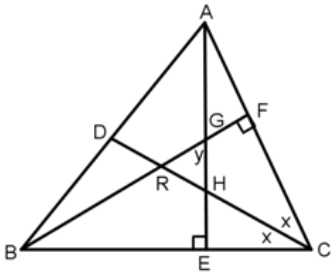


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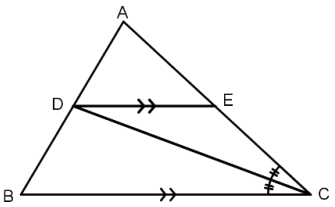
(6) Given the information in the diagram:
(a) Find $\angle BAC, \angle BDC$ in terms of a, b.
(b) If $a = \frac{2}{3}b$, find a and b.



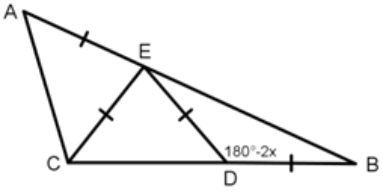
(7) In the following diagram: give an equation that relates x and y.



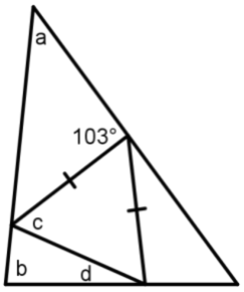
(8) In the following diagram: CD bisects $\angle ACB$, $\angle ACB = 40°$, $\angle B = 70°$, $DE \parallel BC$. Find the measure of $\angle EDC, \angle BDC$.



(9) In $\triangle ABC$, points E and D lie on AB and BC, respectively, such that $AE = EC = DE = DB$.
If $\angle EDB = 180° - 2x$, find $\angle ACE$ in terms of x.



(10) In the following figure:
If $a = \frac{2}{3}c = \frac{1}{2}b = 3d$, find a, b, c, d.



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Triangle Congruence

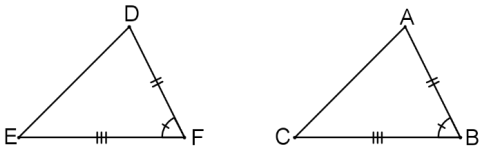
If we draw two triangles, let's say the side lengths of the first are $4, 5, 7$, and then we draw the second triangle using the exact same side lengths on a transparent sheet.

Placing the transparent sheet (which holds the second triangle) directly over the sheet containing the first triangle, we would observe that the two triangles perfectly coincide (superpose). We would also notice that all of the corresponding angles are equal in measure.

This phenomenon is what we call triangle congruence. The example mentioned here illustrates one specific case of congruence, and we will now proceed to discuss the cases of congruence in detail.

Cases of Triangle Congruence

First Case:
Two triangles are congruent if two sides and the included angle of the first triangle are congruent (equal in measure) to the corresponding two sides and the included angle of the second triangle. We will refer to this case of congruence using the abbreviation SAS (side-angle-side).



In triangles $\triangle ABC, \triangle DFE$, if: $AB = DF$, $BC = FE$, $\angle B \cong \angle F$

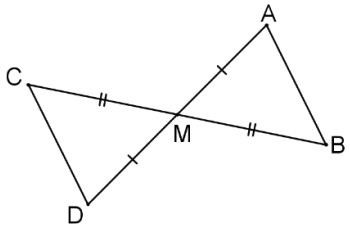
Then $\triangle ABC \cong \triangle DFE \quad (SAS)$

And thus: $AC = DE$, $\angle A \cong \angle D$, $\angle C \cong \angle E$

Exercise (1):

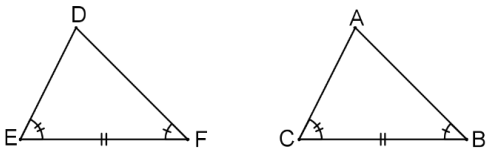
In the following figure, M is the midpoint of AD and BC. Show that:

$$AB = CD\,, \qquad AB \parallel CD$$



Second Case:
Two triangles are congruent if two angles and the included side (the side connecting the vertices of the two angles) in one triangle are congruent (equal in measure and length) to the corresponding two angles and included side in the other triangle. We will call this ASA (angle-side-angle).

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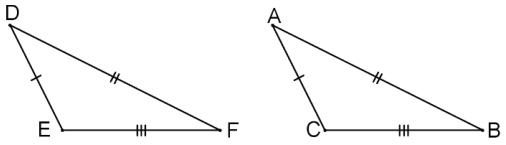


In triangles ABC, DFE, if: $CB = EF$, $\angle C \cong \angle E$, $\angle B \cong \angle F$

Then $\triangle ABC \cong \triangle DFE \quad (ASA)$

Moreover, we get that: $AB = DF$, $AC = DE$, $\angle A \cong \angle D$

Third Case:
Two triangles are congruent if all three sides of the first triangle are congruent (equal in length) to the corresponding three sides of the second triangle. We will refer to this by SSS.

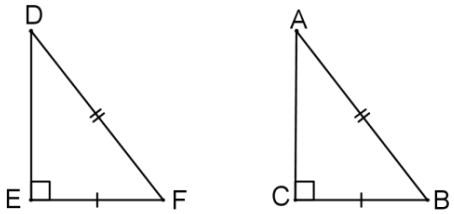


In triangles ABC, DFE, if: $AB = DF$, $AC = DE$, $BC = EF$

Then $\triangle ABC \cong \triangle DFE \quad (SSS)$.

Moreover, we get that: $\angle A \cong \angle D$, $\angle C \cong \angle E$, $\angle B \cong \angle F$

Fourth Case:
Two right-angled triangles are congruent if the hypotenuse and one leg (side forming the right angle) of the first triangle are congruent (equal in length) to the corresponding hypotenuse and one leg of the second triangle. We will refer to this by HS (hypotenuse-side).



In triangles ABC, DFE, if: $AB = DF$, $CB = EF$, $m(\angle C) = m(\angle E) = 90°$

Then $\triangle ABC \cong \triangle DFE \quad (HS)$,

Moreover, we get that: $AC = DE$, $\angle A \cong \angle D$, $\angle B \cong \angle F$

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Remarks:
The two congruent sides of an isosceles triangle are called the legs, and the third side is called the base. The two angles at the base are called the base angles, and the angle opposite the base is called the vertex angle.

Isosceles Triangle Theorem:
The base angles of an isosceles triangle are equal. Conversely, if two angles of a triangle are equal, then the triangle is isosceles.

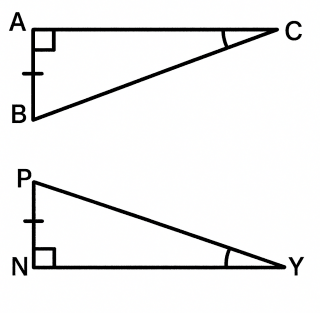
Conclusions:
The angles of an equilateral triangle are all equal, and each measures $60°$.
The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.
The triangle with equal angles is also an equilateral triangle (has equal sides).

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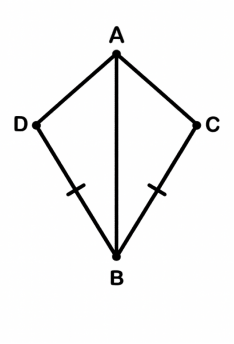
Exercises:

In the following figures, find (if it exists) the triangle that is congruent to $\triangle ABC$ and state the postulate you used.

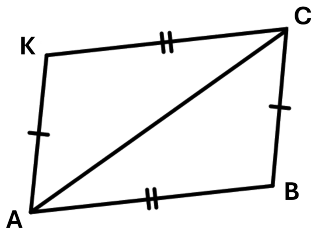
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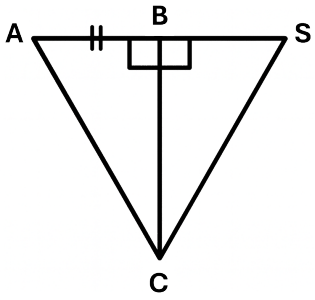
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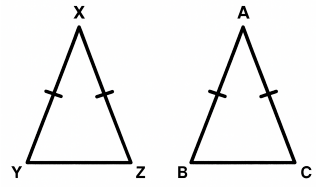
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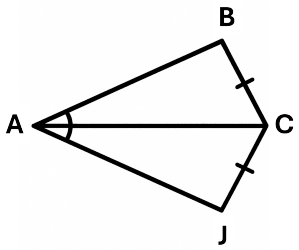
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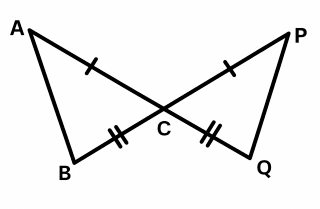
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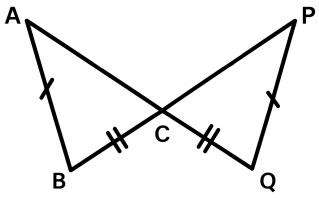
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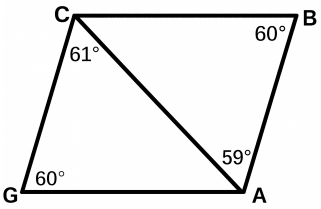
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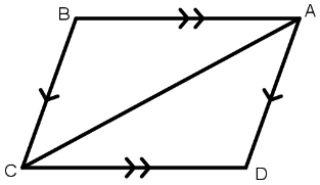


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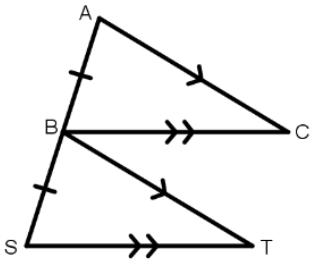


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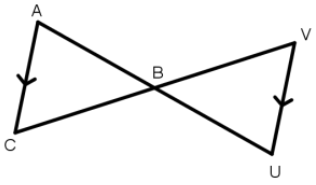
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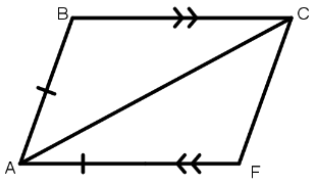
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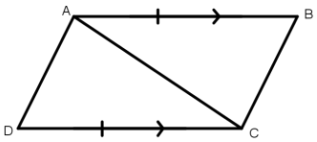
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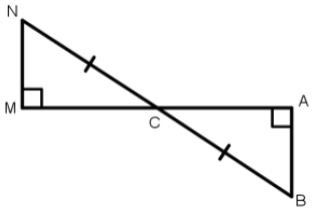
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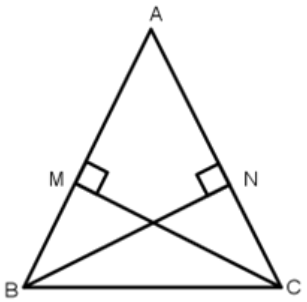
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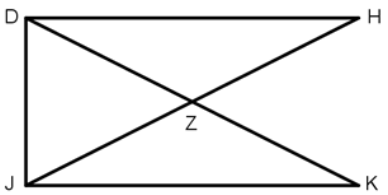
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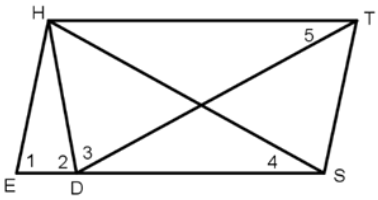
(16) In the adjacent figure: If $BN \perp AC$, $CM \perp AB$, $AB = AC$, explain how you can prove that $\triangle ABN \cong \triangle ACM$.



(17) In the adjacent figure:
If $JH = DK, DH \perp DJ, JK \perp DJ$, prove that $\angle H = \angle K$.

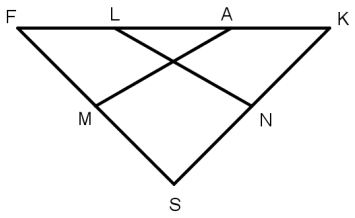


(18) In the adjacent figure:
If $ES = DT$, $\angle 1 = \angle 2 = \angle 3$,
prove that $\angle 4 = \angle 5$.

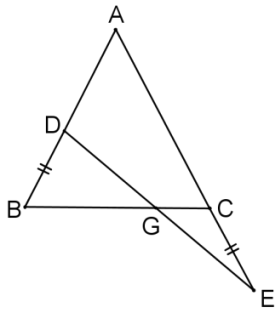


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(19) In the adjacent figure:
If $FL = AK, SF = SK$,
and M is the midpoint of SF, N is the midpoint of SK,
prove that $AM = LN$.



(20) In the triangle $\triangle ABC$, $AB = AC$, point D lies on AB, and point E lies on the extension of AC such that $BD = CE$.
If DE intersects BC at G, prove that $DG = GE$.



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Third Unit: Number Theory



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Even and Odd Integers

Integers can be divided into two types of numbers: those that are divisible by 2 are called **even** numbers, and the rest of the numbers that are not divisible by 2 are called **odd** numbers. An even integer can be written in the form $2k$, where k is an integer. An odd integer can also be written in the form $2k + 1$, where k is an integer. Any given integer is exclusively either even or odd; it cannot be both.

Other Properties of Even and Odd Numbers:

(1) odd $\neq$ even.
(2) odd $+$ even $=$ even $+$ odd $=$ odd, odd $-$ even $=$ even $-$ odd $=$ odd.
(3) even $\pm$ even $=$ odd $\pm$ odd $=$ even.
(4) If the product of two integers is even, then one of them has to be even.
(5) The product of two consecutive integers has to be even.
(6) If the sum (or difference) of some integers is odd, then the number of odds in those numbers is odd.
(7) If the sum (or difference) of some integers is even, then the number of odds in those numbers is even.
(8) If the product of some integers is odd, then all of those integers must be odd.
(9) If the product of some integers is even, then at least one of those integers must be even.

Example 1:

Divide the following numbers into even and odd: $102,\ 203,\ 519,\ 3340,\ 70015,\ 87654$. Then find a pattern for even and odd numbers.

Solution:

The number 102 is even since we can write it as $102 = 2(51)$.

203 is odd since $203 = 2(101) + 1$.

519 is odd since $519 = 2(259) + 1$.

3340 is even since $3340 = 2(1670)$.

Now complete the blanks yourself:

$70015 = 2(35007) + 1$ so it is an ___ number.

$87654 = 2(43827)$ so it is an ___ number.

If the units digit is $0, 2, 4, 6, 8$ then the number is ___.

If the units digit is $1, 3, 5, 7, 9$ then the number is ___.

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Example 2:

If we add two even numbers, then the result would be:

(a) Even (b) Odd

Solution:

The answer is (a), since we know from the properties that the sum of two even numbers is even. Moreover, we can check the units digit: if we add two numbers whose units digits are among $0, 2, 4, 6, 8$, then the resulting number would still have one of $0, 2, 4, 6, 8$ as a units digit.

Can you use the same logic to determine the following:

If we add two odd numbers, then the result would be:

(a) Even (b) Odd

If we add an even and an odd number, then the result would be:

(a) Even (b) Odd

Example 3:

If we pick three random integers a, b, c then the numbers:

$$\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$$

(a) All are non-integers. (b) At least one of them is an integer.

(c) At least two of them are integers. (d) All of them are integers.

Solution:

The answer is (b). There are two cases:

Case 1: All of the numbers have the same parity (all odd or all even), which means that the sum of any two of them is even, so it is divisible by 2. This leads to $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ being all integers.

Case 2: One of them has a different parity (one odd and two even or one even and two odd). For example, if a and c are odd and b is even, then $a+b$ and $b+c$ are odd while $a+c$ is even. Thus, only one of $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ is an integer. Following the same logic, the other case also results in exactly one integer, which gives the answer (b).

Example 4:

If you know that the sum of 100 positive integers is 10000, and that the number of odd numbers (in the 100 numbers) is more than the number of even numbers, find the largest number of even numbers in the set.

Solution:

Let x be the number of even numbers and y the number of odd numbers. Then:

$$x + y = 100, y > x$$

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Since we know that the sum is even, y has to be even (see the properties of even and odd numbers). Thus, y is at least 52, and that means that x is at most 48. It suffices to give an example which is:

$$\underbrace{1+1+\cdots+1}_{52 \text{ times}}+\underbrace{2+2+\cdots+2}_{47 \text{ times}}+9854=10000$$

Thus, the answer is 48.

Example 5:

We have n numbers $x_1, x_2, x_3, \ldots, x_n$, each of them equal to 1 or -1. If:

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$$

Then what can we say about n?

(a) even (b) odd (c) n is a multiple of 4 (d) cannot say anything about n.

Solution:

The answer is (c). Notice that the sum:

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$$

contains n terms, each of which is either 1 or -1. Thus, n has to be even so that the sum is 0. Write $n = 2k$; then k of the numbers $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1$ are equal to 1 and the other k are equal to -1. But notice that their product is:

$$(x_1x_2)\cdot(x_2x_3)\cdots(x_nx_1) = x_1^2x_2^2\cdots x_n^2 = 1$$

Thus:

$$(-1)^k(1)^k = 1$$

This means that k should also be even. Therefore, $k = 2m \Rightarrow n = 2k = 4m$ which gives us that n is a multiple of 4.

Note that the converse is also true: for every integer $n = 4m$ there are numbers $x_1, x_2, \ldots, x_n$, each equal to 1 or -1, that satisfy $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$. For example, take:

$x_1 = x_4 = x_5 = x_8 = \cdots = x_{4m-3} = x_{4m} = 1$,

$x_2 = x_3 = x_6 = x_7 = \cdots = x_{4m-2} = x_{4m-1} = -1$.

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Exercises:

(1) Find the parity of the number $\frac{12214509875435678886333454214938}{2}$ without finding the value of the number.

(2) We have 50 books that we would like to put in 5 boxes. Can we divide the books into the boxes so that each box has an odd number of books? Explain your answer.

(3) Let a, b be two consecutive integers, $c = ab$ and $N^2 = a^2 + b^2 + c^2$. Determine the parity of the number N (odd or even) and show your reasoning.

(4) Challenge: in the following pattern $1, 2, 5, 13, 34, 89, \ldots$ in any three consecutive numbers, the sum of the first and third numbers is equal to three times the number in the middle. For example, $1, 2, 5$ we have $1 + 5 = 3(2)$. Or $2, 5, 13$ we have $2 + 13 = 3(5)$ and so on. What is the parity of the 2003^{rd} number?

(5) Three integers x, y, z are given; two of them are odd and the third is even. Show that

$$(x + 1)(y + 2)(z + 3)$$

is even.

(6) (a) We have the statement:

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 = 0$$

Can we replace the stars with $+, -$ to make the statement correct?
(b) Same question for the statement:

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 = 0$$

(7) In a chess competition, each player will play 20 games. Wins add 3 points to the total score, draws add one point to the score, and losses subtract one point from the total score. Can a player get a final score of 39? Show your reasoning.

(8) (a) There are 10 baskets on a circle. Can we put oranges in each basket so that the difference in the number of oranges between consecutive baskets is 1?
(b) What if there were 9 baskets instead? Show your reasoning.

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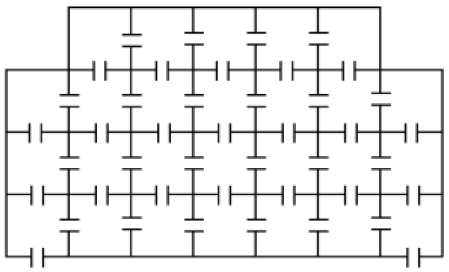
(9) A worm moves on a straight line; it can jump 6 or 8 cm in either direction (left or right). Can the worm reach a point that is:

(a) 1.5 cm (b) 7 cm (c) 4 cm

away from its original position?

(10) Challenge: Can we arrange the 10 numbers $1, 1, 2, 2, 3, 3, 4, 4, 5, 5$ in a line so that there is one number between $1, 1$, two numbers between $2, 2$, three numbers between $3, 3$, four numbers between $4, 4$, and 5 numbers between $5, 5$? Justify your answer.

(11) Challenge: Can we visit the 26 rooms in the diagram so that we enter each room exactly one time? Justify your answer.



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Divisibility

Definition:

> We say a number a divides b if and only if b can be written as $a \cdot k$ where k is an integer. Formally we write it as $a \mid b$ and read it a divides b or b is divisible by a.

For example:

$2 \mid 10$ because $10 = 2 \times 5$

$3 \mid -27$ because $-27 = 3 \times (-9)$

$6 \mid 0$ because $0 = 6 \times 0$

$6 \nmid 10$ because there is no integer k with $6k = 10$

Properties of divisibility:

> $a \mid b$ if and only if $b = ak$ for some integer k.
> If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.
> If $a \mid b$ and $a \mid c$, then $a \mid (b - c)$.
> If $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m, n.
> If $a \mid b$ and $b \mid c$, then $a \mid c$.
> If $a \mid b$, then $a \mid bc$ for every integer c.
> If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Example 1:

Find all positive integer values of n that satisfy $n - 1 \mid 3n + 1$

Solution:

$n = 2, 3, 5$

Note that $n-1 \mid 3(n-1) = 3n-3$ and by the subtraction property we get $n-1 \mid 3n+1-(3n-3) = 4$ and divisors of 4 are $1, 2, 4$ (we reject the negative since n is positive) hence $n - 1 = 1, 2,$ or 4, so $n = 2, 3,$ or 5.

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Exercises:

(1) Prove the properties of divisibility using the definition.

(2) Suppose that: $11 \mid (a+3)$ and $11 \mid (b-4)$. What is the remainder of $b-a$ when divided by 11?

(3) We know that: $n \mid 100$ and $n \mid 36$. What are all possible positive values of n?

(4) Suppose that $n \mid 2a+3b$ and $n \mid 5a+7b$. Show that $n \mid a$ and that $n \mid b$.

(5) If $a \mid b$ and $a+b=100$, find all possible positive values of a and b.

(6) Find all positive integers n such that: $(n^2+5n+11)/(n+2)$ is an integer.

(7) Show that $6 \mid n^3-n$ for any positive integer n.

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Fourth Unit: Combinatorics



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The Two Fundamental Counting Principles

Let us begin with this very simple example.

Example:

Mohammed and Ali went to a sportswear store that sells four different types of shoes and seven different types of sports gloves. Mohammed has enough money to buy one pair of shoes and one pair of gloves, while Ali has enough money to buy only a pair of shoes or only a pair of gloves. The question:

(a) In how many different ways can Mohammed buy a pair of shoes and a pair of gloves?

(b) In how many different ways can Ali buy a pair of shoes or a pair of gloves, but not both together?

Solution:

(a) $4 \times 7 = 28$ ways.

(b) $4 + 7 = 11$ ways.

Many counting problems use one of these two principles, which is why they were named the two fundamental counting principles. Let us get to know them and see the creative ways of applying them in counting problems.

> **The Multiplication Principle:** If event A can occur in m different ways and event B can occur in n different ways, and the two events are independent, then the two events can occur together in $m \cdot n$ ways.

The idea of the Multiplication Principle lies in the way the total number of possible outcomes is computed. Event A can occur in m ways, and for each possible outcome among these outcomes, event B has n possible outcomes. Since each case of A gives us n outcomes, in total the number of possible outcomes is $n + n + \ldots + n = m \times n$. This idea can be generalized to any number of events, as long as the number of possible outcomes of each of them does not depend on the outcome of any of the other events (this is what is meant by independent events in the definition above).

Examples:

In how many ways can we arrange 5 different books on a shelf?

Solution:

First place	Second place	Third place	Fourth place	Fifth place
5 ways	4 ways	3 ways	2 ways	One way

When choosing a book to put in the first place at the beginning of the shelf, we have 5 choices because we have 5 different books. In each of these 5 cases, after we choose one of the books and put it in the first place, we have 4 ways to choose the book that we will put in the second place, regardless of which book we chose at the beginning, because the number of books decreased after

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one of them was put in the first place. In the same way, we count the number of choices for each of the five places and then use the Multiplication Principle to find the number of arrangements, so the number of ways = $5 \times 4 \times 3 \times 2 \times 1$, which equals 120 ways.

Example:

In how many ways can we choose 4 cards that are different in color and different in value (the number written on them), if we have 52 cards divided into 4 colors, and the cards of each color are numbered from 1 to 13?

Solution:

We are required to draw one card of each color, each carrying a number different from those on the cards of the other colors.

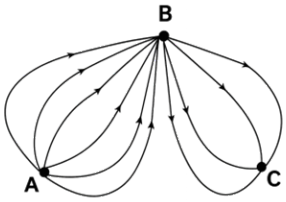
First color	Second color	Third color	Fourth color
13	12	11	10

We have 13 choices for the number of the card of the first color. When choosing a card of the second color, we must exclude the card that bears the same number as the one on the first card, so the number of choices decreases and becomes 12. In the same way, we have 11 ways to choose the card of the third color and 10 ways to choose the card of the fourth color; then we use the Multiplication Principle to find the number of ways, so the number of ways = $13 \times 12 \times 11 \times 10 = 17160$.

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Exercises:

(1) There are three villages A, B, C in a country. One can travel from village A to village B by 6 roads, and one can go from village B to village C by 4 roads, as shown in the figure. In how many different ways can one travel from A to C?



(2) How many three-digit natural numbers are there?

(3) In how many ways can 5 students be arranged in a row if they are chosen from 8 students?

(4) In how many ways can one enter a mall through one door and exit through another door, given that the mall has ten doors?

(5) We want to arrange 4 girls and 5 mothers in a row provided that the girls stay next to one another. In how many ways can this be done?

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> **The Addition Principle:** If event A can occur in m different ways and event B can occur in n different ways, and the two events are mutually exclusive, then event A or B can occur in $m+n$ different ways.

The difference between the Addition Principle and the Multiplication Principle is that if the two events A, B are both going to occur, then the total number of possibilities is the number of ways B can occur for each way A can occur, which leads to the Multiplication Principle. But if the two events A, B are mutually exclusive, that is, we must choose one of them, then in this case either A occurs in m ways or B occurs in n ways, so the total number of ways becomes $m+n$. The Addition Principle is usually used to divide a complicated problem into a number of cases that are easy to deal with.

Example:

We have 5 different books on a shelf. In how many ways can we arrange some (or all) of the books in an ordered group of books, where the ordered group may contain one book or more?

Solution:

We have several cases:

Case 1: The ordered group consists of one book, so the number of ways = 5 ways.

Case 2: The group consists of two books, so the number of ways $= 5 \times 4 = 20$ ways.

Case 3: The group consists of three books, so the number of ways $= 5 \times 4 \times 3 = 60$ ways.

Case 4: The group consists of four books, so the number of ways $= 5 \times 4 \times 3 \times 2 = 120$ ways.

Case 5: The group consists of five books, so the number of ways $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Using the Addition Principle over the cases, the number of ways = $5 + 20 + 60 + 120 + 120$, which equals 325 ways.

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Exercises:

(6) In how many ways can we choose a pen or a notebook from a collection that contains 10 different notebooks and 7 different pens?

(7) Salma has tea cups in 5 different colors, serving plates in 3 different colors, and teaspoons in 4 different colors. In how many different ways can she choose two pieces of two different types of tableware?

(8) A first-grade class has 12 girls, three of whom have names that begin with the letter A. In how many ways can 4 of these girls be arranged in a row provided that only one of the girls has a name that begins with the letter A?

(9) Al-Mawahib School has three classes: the first class consists of 20 students, the second class consists of 13 students, and the third class consists of 8 students. In how many different ways can two students from two different classes be chosen to carry out a task together?

(10) One can travel from city A to city B using one of two land routes or using one of three air routes, and one can travel from city B to city C by one of four land routes or one of five air routes.
(a) How many ways are there to travel from city A to city C passing through city B?
(b) How many ways are there to travel from city A to city C passing through city B, provided that you use two air routes?
(c) How many ways are there to travel from city A to city C passing through city B, provided that you use two land routes?
(d) How many ways are there to travel from city A to city C passing through city B, provided that you use one land route and one air route?

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Counting Numbers and Strings

This is a common type of problem that can be solved using the counting principles, in which we are asked to find how many numbers satisfy a certain condition. To clarify the upcoming problems, you may need these definitions:

> **A natural number of n digits:**
> We say that a natural number consists of n places (digits) if its highest place (the first digit from the left and the n-th from the right) is nonzero.

For example: the number 2345 consists of four places, but we do not consider the number 034 to consist of three places, because the zero in the highest place does not count, and the number is written in the form 34; that is, the number consists of only two places.

> **A string of n digits:**
> We say that a string consists of n places regardless of whether its highest place is zero or not.

For example: each of the following, 01234, 00034, and 12302, is a string consisting of five places.

> **Palindromic numbers:**
> We say that a natural number is palindromic if reading the digits of the number from right to left is identical to reading the digits of the number from left to right.

For example: the numbers 1234321, 919, and 6 are palindromic numbers, while the number 2342 is not palindromic.

The two counting principles will help us count the required numbers or strings in a simple yet deep way. Let us begin the journey of creativity!

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Exercises:

(11) How many positive integers consist of four different digits?

(12) How many positive odd integers consist of five digits?

(13) How many two-digit numbers can be formed from the digits of the set $\{0, 3, 4, 5, 6, 7, 8, 9\}$, provided that the sum of the digits is odd?

(14) How many numbers between 0 and 1000 contain the digit 5 exactly once?

(15) How many numbers that are divisible by 3 and have 3 different digits can be formed from the digits of the set $\{1, 3, 7, 8, 9\}$?

(16) How many strings of 10 digits, consisting only of ones and zeros, contain exactly five adjacent zeros?

(17) How many five-digit numbers satisfy the condition that the sum of the first digit and the fifth digit equals 5?

(18) How many six-digit numbers have all of their digits of the same type (all even or all odd)?

(19) How many positive integers with 4 digits have exactly one digit "1" and exactly one digit "3"?

(20) How many numbers consisting of 3 digits have at least one even digit?

(21) How many four-digit numbers are there in which no two even digits are adjacent?

(22) How many positive integers less than 1000 do not contain the digit 7?

(23) How many positive integers consisting of five different digits satisfy the condition that the positive difference between the first digit and the last digit equals 2?

(24) When writing the numbers from 1 to 100, how many times will we write the digit 6?

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(25) How many palindromic numbers are there with:
(a) 6 digits from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$?
(b) 7 digits from the same set?
(c) 5 digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$?

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Counting Words

The number of English words consisting of n letters can be computed directly using the Multiplication Principle. In counting problems, the resulting words do not necessarily have to have a meaning.

Example:

The number of words consisting of 4 letters equals

First letter		Second letter		Third letter		Fourth letter	Number of words
Any letter		Any letter		Any letter		Any letter	
26 ways	$\times$	26 ways	$\times$	26 ways	$\times$	26 ways $=$	26^4

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Exercises:

(26) How many words consist of 5 different letters?

(27) How many ways are there to arrange the letters of the word PRODUCT?

(28) How many words consist of 6 letters, provided that the first, third, and sixth letters are different?

(29) How many words consisting of 3 letters contain the letter M exactly once?

(30) How many words consisting of 3 letters contain the letter M at least once, where repeating any other letter is not allowed?

(31) How many words consisting of 3 letters contain the letter M at least once, where repeating another letter is allowed (another letter may appear twice)?

(32) The Saudi team for the Cybersecurity Olympiad invented a language of its own consisting of only four letters. In their language, the longest word contains at most 6 letters. How many words does this language have?

(33) We define a good word as a word consisting only of the letters A, B, and C (any of these letters may be absent from the word) such that the letter B never immediately follows the letter A, the letter C never immediately follows the letter B, and the letter A never immediately follows the letter C. How many good words consist of 7 letters?

(34) How many palindromic words consisting of seven letters do not begin with a vowel? The vowels are (A, E, I, O, U).

(35) How many ways are there to write all the English letters in a row such that there are exactly five letters between the two letters x and y?

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Answers



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A note to the student
We believe that learning mathematics is about thinking hard about a problem and getting stuck — and pushing through. That is how real problem-solving ability is built. This is why this section gives only the final answers, not the full solutions: try every problem on your own first. If you want to check the full solutions, watch the NSMO videos.

Algebra Answers

Linear Equations in One Variable:

(1) $(a)\ x = 18$; $(b)\ x = 10$; $(c)\ x = 5$; $(d)\ x = 8$

(2) $x = 1$

(3) $x = 3$

(4) $(a)\ 800$; $(b)\ 14$; $(c)\ 28$; $(d)\ 2000$ riyals; $(e)\ 40000$ riyals

Challenge Problems (1):

(1) The train is 60 meters long.

(2) $6 : 5$

(3) 33 years old.

(4) 24 oranges.

(5) The seventh number is 11.

(6) Since $17 < 20$ and $481 < 500$, the true price is less than $20 \times 500 = 10{,}000$ riyals, so it cannot be $10{,}177$ riyals.

(7) No; the result is always even, while 20042401 is odd.

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(8) $983 + 75 + 4 = 1062$

---

(9) Construction problem — see the NSMO videos.

---

(10) 16 liters per day.

---

(11) Construction problem — see the NSMO videos.

---

(12) $JMO = 150,\ IMO = 450$ or $JMO = 250,\ IMO = 750$

Multi-Variable Linear Equations:

(1) The system has infinitely many solutions.

---

(2) $(a)\ x = 10,\ y = 3$; $(b)\ x = 4,\ y = 1$; $(c)\ x = 8,\ y = 6$; $(d)\ x = 5,\ y = 4$

---

(3) The numbers are 25 and 17.

---

(4) It was $3{:}09$.

---

(5) $x = \frac{59}{4},\ y = -\frac{11}{4}$

---

(6) $x = 15,\ y = 5$

---

(7) $x = 2,\ y = -3$

---

(8) $x = \frac{30}{41},\ y = \frac{45}{41},\ z = \frac{75}{41}$

---

(9) $x = 1,\ y = 2,\ z = 3,\ u = 4$

---

(10) $x = \frac{2a+b-c}{14},\ y = \frac{2b+c-a}{14},\ z = \frac{2c+a-b}{14}$

---

(11) $4a - c = 9$

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(12) $(a, b, c) = (2, 3, 4)$

---

(13) (a) $k \neq -2$ (with solution $x = 0,\ y = \frac{1}{3}$); (b) no value of k; (c) $k = -2$

---

(14) $z - y = 2021$

---

(15) $x = 7,\ y = 5,\ z = -3$

---

(16) $x = 0,\ y = 6,\ z = 7,\ u = 3,\ v = -1$

Challenge Problems (2):

(1) $a = \frac{1}{10},\ b = \frac{1}{4}$

---

(2) $x = 5,\ y = 3,\ z = 1$

---

(3) E.g. $\frac{1}{2}$ and $\frac{1}{3}$ (also any pair $\frac{1}{a}$ and $\frac{1}{a+1}$).

---

(4) $3 : 7$

---

(5) 23 children.

---

(6) $c = 1$

---

(7) 553451234512345

---

(8) Construction problem — see the NSMO videos.

---

(9) Nour is a girl.

---

(10) 25 liars.

---

(11) No; at most 16 squares can be colored.

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Percentages:

(1) $k = 18$

(2) $a = 5,\ b = 2,\ c = 1$

(3) 30

(4) 30%

(5) 18 cookies.

(6) 162°

(7) $a : c = 17 : 21$

(8) 80%

(9) 3125 fish.

(10) $4t : 2u = 1 : 2$

(11) $k = \frac{1}{4}$

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Geometry Answers

Revision Exercises:

(1) $4n + m = 180°$

(2) $\angle A = 23°$

(3) $\angle EAD = 141°$

(4) $x = a - b + c + d$

(5) $\angle DBF = 36°$

(6) $(a)\ \angle BAC = 2b - 2a$, $\angle BDC = b - a$; $(b)\ a = 39°$, $b = 58.5°$

(7) $y = 2x$

(8) $\angle EDC = 20°$; $\angle BDC = 90°$

(9) $\angle ACE = \frac{180° - 3x}{2}$

(10) $a = 42°$, $b = 84°$, $c = 63°$, $d = 14°$

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Triangle Congruence Exercises:

Exercise (1): Proof problem — see the NSMO videos.

Ex. No	Congruence Type	Congruent Triangles
1	ASA	$ABC \cong NPY$
2	None	None
3	SSS	$ABC \cong CKA$
4	None	None
5	None	None
6	None	None
7	SAS	$ABC \cong PQC$
8	None	None
9	ASA	$ABC \cong AGC$
10	ASA	$ABC \cong CDA$
11	ASA	$ABC \cong BST$
12	None	None
13	None	None
14	SAS	$ABC \cong CDA$
15	ASA	$ABC \cong MNC$

(16) Proof problem — see the NSMO videos.

(17) Proof problem — see the NSMO videos.

(18) Proof problem — see the NSMO videos.

(19) Proof problem — see the NSMO videos.

(20) Proof problem — see the NSMO videos.

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Number Theory Answers

Even and Odd Integers:

(1) The number is odd.

(2) No, since the sum of 5 odd numbers is odd while 50 is even.

(3) N is odd.

(4) The 2003^{rd} term is even.

(5) Proof problem — see the NSMO videos.

(6) (a) Yes, e.g. $1+2-3+4+5+6-7-8=0$; (b) Impossible, since $1+2+\cdots+9=45$ is odd.

(7) Impossible: each game scores an odd number of points (3, 1 or -1), so the total of 20 games is even, while 39 is odd.

(8) (a) Yes, e.g. $1,2,1,2,\ldots,1,2$; (b) Impossible, since the parities must alternate around the circle and 9 is odd.

(9) (a) No; (b) No; (c) Yes, e.g. $6+6-8=4$.

(10) Impossible, since the sum of positions under the conditions is even while $1+2+\cdots+10=55$ is odd.

(11) Impossible: with a chessboard coloring there are 14 white and 12 black rooms, and the path alternates between the two colors.

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Divisibility Exercises:

(1) Proof problem — see the NSMO videos.

---

(2) The remainder is 7.

---

(3) $n = 1, 2, 4$.

---

(4) Proof problem — see the NSMO videos.

---

(5) $(a, b) = (1, 99), (2, 98), (4, 96), (5, 95), (10, 90), (20, 80), (25, 75), (50, 50)$.

---

(6) $n = 3$.

---

(7) Proof problem — see the NSMO videos.

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Combinatorics Answers

Counting Principles:

(1) 24 ways

(2) 900 numbers

(3) 6720 ways

(4) 90 ways

(5) 17280 ways

(6) 17 ways

(7) 47 ways

(8) 6048 ways

(9) 524 ways

(10) (a) 45 ways; (b) 15 ways; (c) 8 ways; (d) 22 ways

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Counting Numbers and Strings:

(11) 4536 numbers

(12) 45000 numbers

(13) 28 numbers

(14) 243 numbers

(15) 24 numbers

(16) 64 strings

(17) 5000 numbers

(18) 28125 numbers

(19) 720 numbers

(20) 775 numbers

(21) 4625 numbers

(22) 728 numbers

(23) 5040 numbers

(24) 20 times

(25) (a) 512 numbers; (b) 4096 numbers; (c) 648 numbers

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Counting Words:

(26) $26 \times 25 \times 24 \times 23 \times 22 = 7{,}893{,}600$ words

---

(27) $7! = 5040$ ways

---

(28) $274{,}185{,}600$ words

---

(29) 1875 words

---

(30) 1876 words

---

(31) 1951 words

---

(32) 5460 words

---

(33) $3 \times 2^6 = 192$ good words

---

(34) $21 \times 26^3 = 369{,}096$ words

---

(35) $40 \times 24!$ ways

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