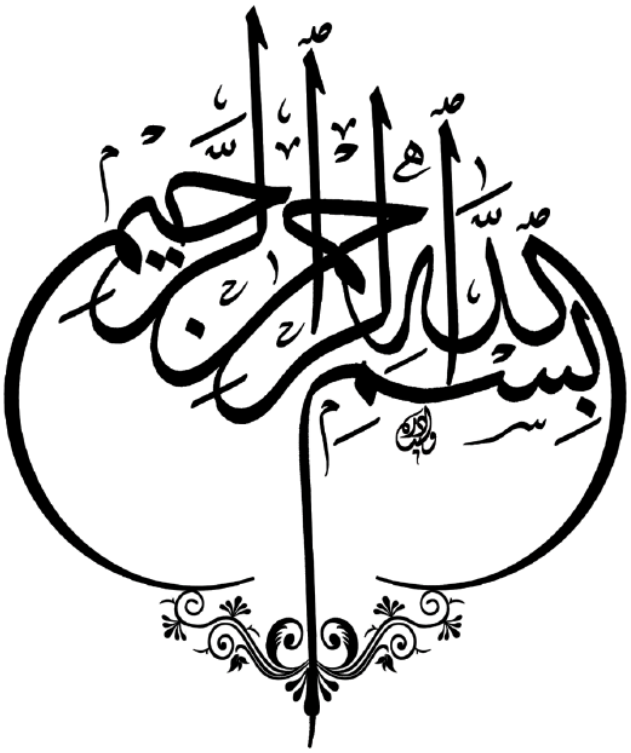
أولمبياد الموهبة الوطني
NATIONAL GIFTED OLYMPIAD

PHASE 2

Scientific Track – NSMO

Senior Mathematics

Mathematics Scientific Team



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Introduction

Our distinguished sons and daughters,

We are pleased to congratulate you on passing Phase One and qualifying for **Phase Two** — an advanced step on the path of mathematical challenge and innovation.

This package is devoted to broadening your horizons in the four main branches: **Combinatorics, Geometry, Algebra, and Number Theory**, with a focus on advanced concepts in counting, geometric review, linear equations, and distribution and multiplication in number theory.

This phase aims to sharpen your analytical thinking skills and to connect mathematical concepts with one another, applying them in a variety of situations.

This package is also an opportunity to deepen your understanding of mathematical patterns and to use logic in justification and in solving problems in organized ways.

We trust in your abilities and look forward to seeing you shine in this important phase of your journey toward excellence.

The Scientific Team of the National Mawhiba Olympiad – Scientific Track
(NSMO) – Mathematics Track

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First Unit: Algebra



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Percentages

Percent literally means "per hundred." Percent is just a shorthand way of writing the ratio of a number to 100.
For example, 54% means 54 out of 100 or $\frac{54}{100}$
Specifically, if a is x percent of b, then

$$\frac{a}{b} = \frac{x}{100}$$

For example, 3 out of 16 is written as a percentage as follows: $\frac{3}{16} = \frac{x}{100}$

By simplifying, we obtain $x = 18\frac{3}{4}$

Final Answer: 3 is 18.75% of 16

If a number x is increased by $k\%$, then the result is:

$$x \cdot \left(1 + \frac{k}{100}\right)$$

Similarly, if x is decreased by $k\%$, then the result is:

$$x \cdot \left(1 - \frac{k}{100}\right)$$

Example:

When a number is reduced by 40%, the result is 36. What is the original number?

Solution:

Let our number be x. We have

$$x\left(1 - \frac{40}{100}\right) = 36$$

$$\Rightarrow \frac{6}{10}x = 36$$

$$\Rightarrow x = 60$$

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Exercises:

(1) 36 is 120% of what number?

(2) What is the discount percentage for an item if its price changes from 2890 riyals to 2023 riyals?

(3) If Nora brought 60 cookies to school, gave 40% of them to her teachers, 25% of the rest to her friends, and ate one-third of what was left, how many cookies are left?

(4) If the ratio of the interior angles of a pentagon is $2:3:4:5:6$, what is the measure of the largest angle?

(5) If b is 5% greater than a, and b is 15% less than c, what is the ratio of a to c?

(6) A rectangle has been increased in length by 50% and in width by 20%. What is the percentage increase in its area?

(7) Every day, 20% of the fish are sold at the fish market. If 2000 fish remained at the end of Tuesday, how many fish were there at the beginning of Monday?

(8) If t is 25% of u, what is the ratio of $4t$ to $2u$?

(9) Given that

$$yz : zx : xy = 1 : 2 : 3 \quad \text{and} \quad \frac{x}{yz} : \frac{y}{zx} = 1 : k,$$

find k.

(10) If the numbers x, y, z satisfy (with all denominators nonzero)

$$\frac{7}{x+y} = \frac{k}{x+z} = \frac{11}{z-y},$$

find the value of k.

(11) Find the values of $a, b, c \in \mathbb{N}$ that satisfy the following:

$$\frac{a+1}{3} = \frac{b+2}{2} = \frac{8}{c+3}$$

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One-Variable Linear Equations

To solve a one-variable linear equation, follow the steps below in order:

Eliminate denominators (if any)

If there are fractions in the equation, multiply each term by the least common multiple (LCM) of the denominators to eliminate them.

Expand parentheses

Use the distributive property to simplify parentheses, such as:

$$2\,(x+3) = 2x + 6$$

Move terms

Move all the terms with the variable to one side and all the constants to the other using addition and subtraction.

Combine like terms

Simplify both sides of the equation so that it takes the form:

$$ax = b, \text{ where } a \text{ and } b \text{ are constants.}$$

Divide by the variable's coefficient

We divide both sides by a (the coefficient of the variable) to get the solution:

$$ax = b \Rightarrow x = \frac{b}{a}$$

Note that:

> When $a \neq 0$ we have a unique solution:
>
> $$x = \frac{b}{a}$$
>
> When $a = 0$ and $b \neq 0$, there is no solution.
> When $a = 0$ and $b = 0$, every real number is a solution of the equation.

Check your answer

We can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

Note: Sometimes we do not follow the exact order of the previous steps when a more direct method is available.

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Example:

Solve:

$$\frac{1}{10}\left\{\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8\right\}=1$$

Solution:

$$\frac{1}{10}\left\{\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8\right\}=1$$

Multiplying both sides by 10:

$$\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]+8=10$$

Subtracting 8 from both sides:

$$\frac{1}{9}\left[\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16\right]=2$$

Multiplying both sides by 9:

$$\frac{1}{5}\left(\frac{x+2}{3}+8\right)+16=18$$

Subtracting 16 from both sides:

$$\frac{1}{5}\left(\frac{x+2}{3}+8\right)=2$$

Multiplying both sides by 5:

$$\frac{x+2}{3}+8=10$$

Subtracting 8 from both sides:

$$\frac{x+2}{3}=2$$

Multiplying both sides by 3:

$$x+2=6$$

Subtracting 2 from both sides:

$$x=4$$

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Exercises:

(1) Solve:

$$1-\frac{x-\frac{1+3x}{5}}{3}=\frac{x}{2}-\frac{2x-\frac{10-6x}{7}}{2}$$

(2) If a, b, c are fixed positive numbers, solve the equation:

$$\frac{x-a-b}{c}+\frac{x-b-c}{a}+\frac{x-c-a}{b}=3$$

(3) Solve:

$$(x-3)^2+(x+1)^2+(4x-5)^2=0$$

(4) Solve:

$$ax+b-\frac{5x+2ab}{5}=\frac{1}{4}$$

(5) Given that the equation

$$a(2x+3)+3bx=12x+5$$

has infinitely many solutions for x, find the values of a and b.

(6) Given that the equation

$$2a(x+6)=4x+1$$

has no solution, where a is a parameter, find the value of a.

(7) Given that the equation

$$kx=12$$

has only positive integer solutions, where k is an integer, find the number of possible values of k.

(8) How many positive integers x satisfy the following equation?

$$\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}=\frac{13}{12}$$

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(9) Given that the solution of the equation

$$3a - x = \frac{x}{2} + 3$$

is 4, find the value of $(-a)^2 - 2a$.

(10) Solve the equation

$$\frac{x-n}{m} - \frac{x-m}{n} = \frac{m}{n}$$

(where $mn \neq 0$ and $m \neq n$).

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Multi-Variable Linear Equations

> A system of linear equations consists of two or more equations that share the same unknown variables (in the simplest case, two equations in two unknowns, usually x and y).

The goal is to find the pairs of numbers (x, y) that make both equations true at the same time. You can also think about the equations as two straight lines in the coordinate plane, and our goal is to find the coordinates of their intersection.

The Elimination Algorithm

The most reliable and straightforward way to solve these systems is the **Elimination Method**. The goal is to manipulate the equations so that when you add them together, one of the variables completely disappears.

Example Problem:

Equation 1: $2x + 3y = 12$

Equation 2: $3x - y = 7$

Step 1: Line up the variables.

Ensure both equations are formatted identically. The x variables should be vertically aligned, followed by the y variables, the equals signs, and the constant numbers. (Our example is already correctly aligned).

Step 2: Force a match with opposite signs.

Look at the numbers in front of the y variables: Equation 1 has a positive 3, and Equation 2 has a negative 1. If we multiply everything in Equation 2 by 3, the y terms will perfectly cancel each other out when the equations are added together.

Multiply every term in Equation 2 by 3:

Equation 2 becomes $9x - 3y = 21$

Step 3: Add the equations vertically.

Stack Equation 1 and the newly adjusted Equation 2, then add straight down column by column:

$$2x + 3y = 12$$
$$9x - 3y = 21$$

The addition results in:

$$11x + 0 = 33$$

The y variable is now fully eliminated.

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Step 4: Solve for the remaining variable.

You are left with a basic one-variable equation:

$$11x = 33$$

Divide both sides by 11 to isolate x:

$$x = 3$$

Step 5: Substitute back to find the second variable.

Take your discovered value ($x = 3$) and plug it into either of the original equations to solve for y. Let's use Equation 1:

$$2\,(3) + 3y = 12$$

$$6 + 3y = 12$$

Subtract 6 from both sides:

$$3y = 6$$

Divide by 3:

$$y = 2$$

The final solution to the system is $x = 3, y = 2$.

Determining the Number of Solutions

Instead of memorizing abstract fraction formulas to determine how many solutions a system has, you can simply evaluate the numbers logically.

1. One Unique Solution (Intersecting Lines)

This is the standard scenario. The equations have different variable ratios and will cross at exactly one coordinate point on a graph.

Example: The problem we solved above, which yielded exactly one answer: $x = 3, y = 2$.

2. No Solution (Parallel Lines)

This occurs when the variables in both equations are identical (or perfectly proportional), but the final answers contradict one another. The lines run parallel and never cross.

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Example:

$$\begin{cases} x + y = 5 \\ x + y = 10 \end{cases}$$

It is mathematically impossible for two numbers to add up to both 5 and 10 simultaneously. Therefore, this system has zero solutions.

3. Infinitely Many Solutions (The Same Line)

This happens when the second equation is simply a multiplied or scaled version of the first equation. Because they represent the exact same line on a graph, every point on that line is a valid solution.

Example:

$$\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$$

If you divide the entire second equation by 2, it becomes identical to the first. Any pair of numbers that adds to 5 will successfully solve both equations.

Example:

How many solutions does each of the following systems have?

(a) $\begin{cases} 5x + 2y = 4 \\ 2.5x + y = 2 \end{cases}$ (b) $\begin{cases} 4x + 3y = 3 \\ 8x + 6y = 7 \end{cases}$

Solution:

(a) Infinitely many solutions: the second equation can be obtained by dividing the first by 2, so they represent the same line.

(b) There is no solution: multiplying the first by 2 gives $8x + 6y = 6$, which contradicts the second equation (parallel lines).

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Exercises:

(1) How many solutions are there for this system?

$$\begin{cases} x + 5y = 3 \\ 2x + 10y = 6 \end{cases}$$

(2) Solve the following system of equations:

(a) $\begin{cases} x + y = 13 \\ x - y = 7 \end{cases}$ (b) $\begin{cases} 2x + y = 9 \\ x + 4y = 8 \end{cases}$

(c) $\begin{cases} x + y = 14 \\ x - y = 2 \end{cases}$ (d) $\begin{cases} 2x - y = 6 \\ x + 4y = 21 \end{cases}$

(3) Two numbers have a sum of 42 and a difference of 8. What are the two numbers?

(4) At a certain time, Reema notices that her digital watch reads a minutes after two o'clock. Fifteen minutes later, it reads b minutes after three o'clock. She is amused to note that a is six times b. What time was it when she looked at her watch for the second time?

(5) Solve the following system of equations:

$$\begin{cases} \frac{x-y}{5} - \frac{x+y}{4} = \frac{1}{2}, \\ 2(x - y) - 3(x + y) + 1 = 0 \end{cases}$$

(6) Solve the following system of equations:

$$\begin{cases} 5.4x + 4.6y = 104 \\ 4.6x + 5.4y = 96 \end{cases}$$

(7) Solve the following system of equations:

$$\begin{cases} x + 2\,(5x + y) = 16 \\ 5x + y = 7 \end{cases}$$

(8) Solve the following system of equations:

$$\begin{cases} \frac{x}{2} = \frac{y}{3} = \frac{z}{5} \\ x + 3y + 6z = 15 \end{cases}$$

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(9) Solve the following system of equations:

$$\begin{cases} x+2y=5 \\ y+2z=8 \\ z+2u=11 \\ u+2x=6 \end{cases}$$

(10) Solve the following system of equations:

$$\begin{cases} 5x-y+3z=a \\ 5y-z+3x=b \\ 5z-x+3y=c \end{cases}$$

(11) Given that $x=2, y=1$ is the solution of the system

$$\begin{cases} ax+by=7, \\ bx+cy=5 \end{cases}$$

what is the relation between a and c?

(12) Find the value of (a,b,c) if both systems have the same solution:

$$\begin{cases} 3x-y=5 \\ 2x+y-z=0 \\ 4ax+5by-z=-22 \end{cases} \qquad \begin{cases} ax-by+z=8 \\ x+y+5=c \\ 2x+3y=-4 \end{cases}$$

(13) Determine the values of k such that the system of equations

$$\begin{cases} kx-y=-\frac{1}{3} \\ 3y=1-6x \end{cases}$$

has: (a) a unique solution; (b) no solution; (c) infinitely many solutions.

(14) Given that $x,\ y,\ z$ satisfy the system of equations

$$\begin{cases} 2020(x-y)+2021(y-z)+2022(z-x)=0 \\ 2020^2(x-y)+2021^2(y-z)+2022^2(z-x)=2021 \end{cases}$$

find the value of $z-y$.

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(15) Solve the following system of equations:

$$\begin{cases} x - y - z = 5 \\ y - z - x = 1 \\ z - x - y = -15 \end{cases}$$

(16) Solve the following system of equations:

$$\begin{cases} x - y + z = 1 \\ y - z + u = 2 \\ z - u + v = 3 \\ u - v + x = 4 \\ v - x + y = 5 \end{cases}$$

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Challenge Problems:

(1) Solve the following system of equations:

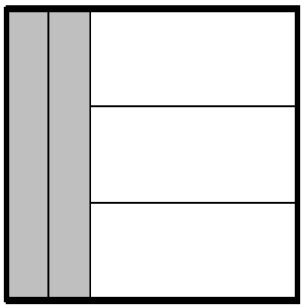
$$\begin{cases} \frac{1}{a} - \frac{1}{b} = 6 \\ \frac{1}{a} + \frac{1}{b} = 14 \end{cases}$$

(2) Solve the following system of equations:

$$\begin{cases} x + y = 8 \\ y + z = 4 \\ z + x = 6 \end{cases}$$

(3) Give an example of two fractions whose difference is equal to their product.

(4) The figure shown is a square composed of five rectangles, all having the same perimeter. What is the ratio of the area of one shaded rectangle to the area of one unshaded rectangle?



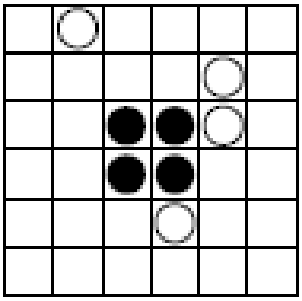
(5) We want to divide a certain amount of money equally among a group of children. If each child receives 60 halalas, 2.10 riyals will remain. However, if the amount of money were 20 halalas more, it would be just enough for each child to receive 70 halalas. How many children are in the group?

(6) The number 3600 can be written as $2^a \times 3^b \times 4^c \times 5^d$, where $a, b, c,$ and d are positive integers. If $a + b + c + d = 7$, what is the value of c?

(7) If you remove ten digits from the number $12345123451234512345123 45$, what is the largest number you can get?

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(8) Cut the following shape into four identical parts (both in shape and area) so that each part contains one black circle and one white circle.

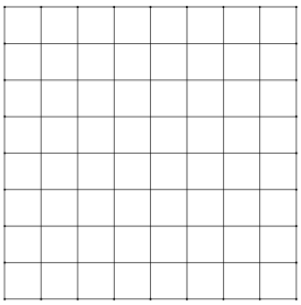


(9) My brother has four children. They are 5, 8, 13, and 15 years old. Their names (in no particular order) are Muhammad, Raja, Najah, and Nour. One of the girls is in kindergarten, Raja is older than Muhammad, and the sum of Raja and Nour's ages is divisible by three.
Is Nour a boy or a girl?

(10) There are one hundred people living on an island. Some of them always lie, while others always tell the truth, and each of them has a favorite season of the year. Every person was asked the following four questions:
- Do you like winter?
- Do you like spring?
- Do you like summer?
- Do you like fall?

There were 25 "yes" answers to the first question, 25 "yes" answers to the second, 45 "yes" answers to the third, and 55 "yes" answers to the fourth. How many liars are there on the island?

(11) We have the following 8×8 table divided into 64 small squares. Can you color 17 small squares black so that no two squares share an edge or even a corner?



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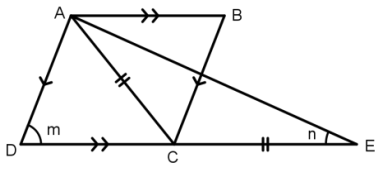
Second Unit: Geometry



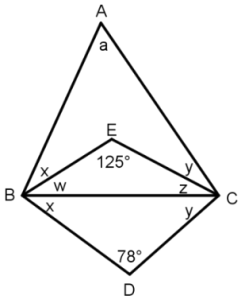
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Revision Exercises:

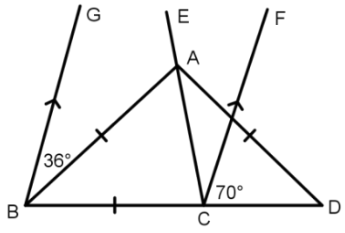
(1) In the following diagram $ABCD$ is a rhombus, ACE is an isosceles triangle with $AC = CE$ and D, E, C are on the same line. If $\angle AEC = n$ and $\angle ADC = m$, write an equation that relates m and n.



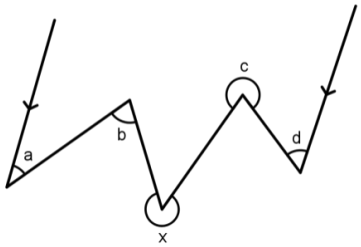
(2) In the adjacent diagram, $ABDC$ is a quadrilateral. Point E lies inside it such that $\angle ABE = \angle DBC = x$ and $\angle ACE = \angle DCB = y$. If $\angle BEC = 125°$ and $\angle BDC = 78°$, find $\angle A$.



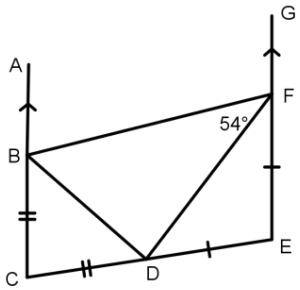
(3) In the following diagram, $AB = BC = AD$. Find $\angle EAD$.



(4) Find the value of x in terms of a, b, c, d.

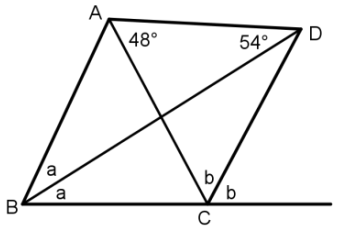


(5) In the following diagram we have $BC = CD$, $DE = EF$ and $AC \parallel GE$.
Given $\angle BFD = 54°$, find $\angle DBF$.

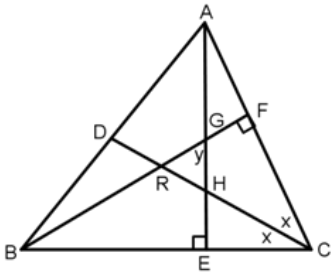


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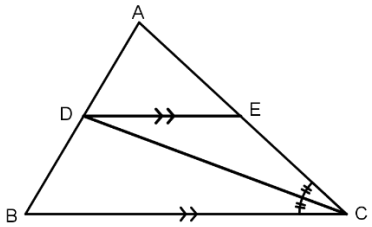
(6) Given the information in the diagram:
(a) Find $\angle BAC, \angle BDC$ in terms of a, b.
(b) If $a = \frac{2}{3}b$, find a and b.



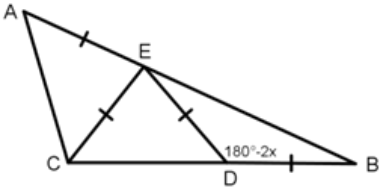
(7) In the following diagram: give an equation that relates x, y.



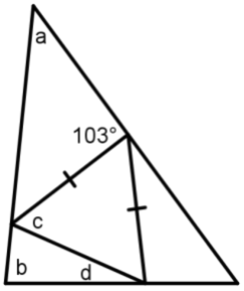
(8) In the following diagram: CD bisects $\angle ACB$, $\angle ACB = 40°$, $\angle B = 70°$ and $DE \parallel BC$. Find the measures of $\angle EDC$ and $\angle BDC$.



(9) In $\triangle ABC$, points E and D lie on AB and BC respectively, such that $AE = EC = DE = DB$. If $\angle EDB = 180° - 2x$, find $\angle ACE$ in terms of x.



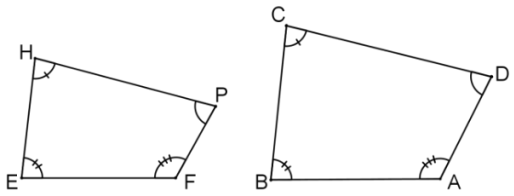
(10) In the following figure, if $a = \frac{2}{3}c = \frac{1}{2}b = 3d$, find a, b, c, d.



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Similarity

Definition 1:



Any two polygons are **similar** if the two following conditions are met:
(1) The corresponding angles are equal in measure.
(2) The lengths of the corresponding sides are proportional.

So, in the polygons above, if we have:

$$\left\{\begin{array}{c} \angle A = \angle F \\ \angle B = \angle E \\ \angle C = \angle H \\ \angle D = \angle P \\ \frac{AB}{FE} = \frac{BC}{EH} = \frac{CD}{HP} = \frac{DA}{PF} \end{array}\right\} \Rightarrow ABCD \sim FEHP$$

Theorem 1:
The ratio between the perimeters of similar polygons is equal to the ratio of any two corresponding sides.

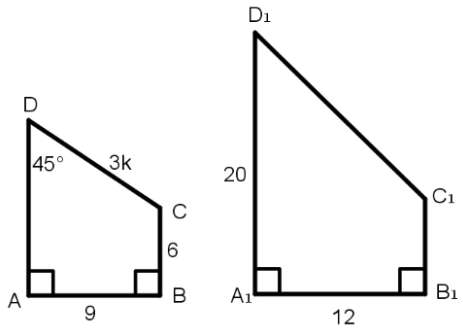
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Exercises:

In exercises (1–8), choose the appropriate answer (only one is correct):

Exercise	Polygons	Always similar	Can be similar	Cannot be similar
(1)	**Equilateral triangles**			
(2)	**Isosceles triangles**			
(3)	**Squares**			
(4)	**Rhombuses**			
(5)	**Right triangles**			
(6)	**Scalene triangles**			
(7)	**Rectangles**			
(8)	**Right triangle and acute triangle**			

Exercises (9–16) refer to the diagram below, where the quadrilaterals $ABCD \sim A_1B_1C_1D_1$ are similar:



(9) Similarity ratio of $ABCD$ to $A_1B_1C_1D_1$

(10) Type of quadrilateral $A_1B_1C_1D_1$

(11) Measure of $\angle D_1$

(12) Measure of $\angle C_1$

(13) Measure of C_1B_1

(14) Measure of AD

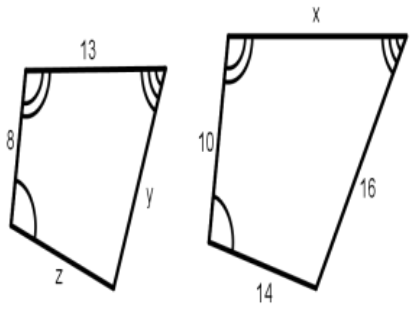
(15) Measure of C_1D_1

(16) Ratio of the perimeters of the quadrilaterals

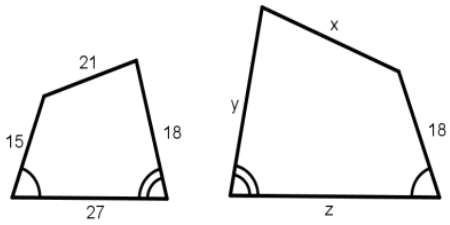
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Find the values of x, y, z if the two polygons in each exercise are similar.

(17)



(18)



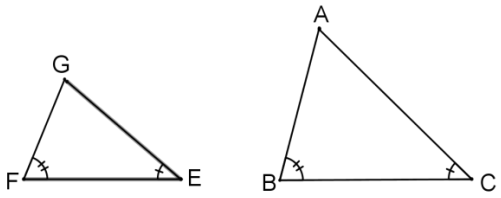
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Triangle Similarity

Theorem 2:
For any two similar triangles we have:
(1) The corresponding sides are proportional.
(2) The corresponding altitudes are proportional and have the same ratio as the similarity ratio.
(3) The corresponding medians are proportional and have the same ratio as the similarity ratio.

Similarity Postulate (AAA)

Theorem 3:
Similarity Postulate: AAA
Two triangles are similar if their corresponding angles are equal in measure. We call this case (AAA) (Angle-Angle-Angle). Of course, it suffices to show similarity using two angles (AA) instead of all three, since the sum of the angles in a triangle is $180°$.



In the adjacent triangles ABC and GFE, if

$$\angle B = \angle F,\ \angle E = \angle C$$

then the triangles are similar and we get that:

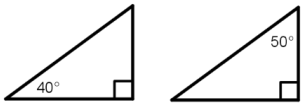
$$\frac{AB}{GF} = \frac{BC}{FE} = \frac{CA}{EG}$$

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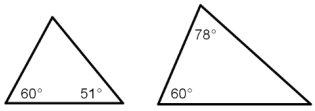
Exercises:

In (1–9), determine the similar triangles; if you cannot reach a conclusion, write “Inconclusive”.

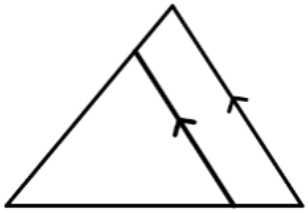
(1)



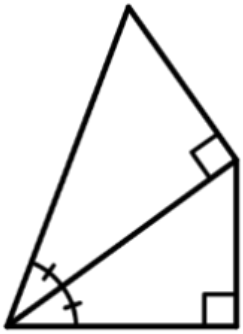
(2)



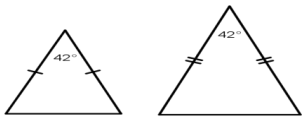
(3)



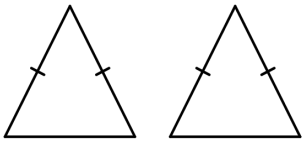
(4)



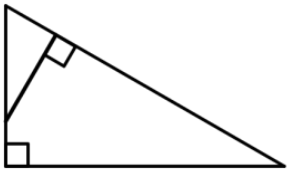
(5)



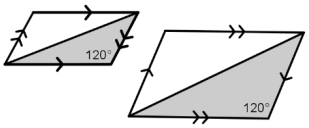
(6)



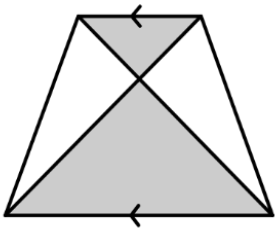
(7)



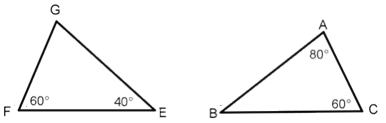
(8)



(9)



(10) Look at the adjacent figure, mark the angles on the two triangles, and complete:

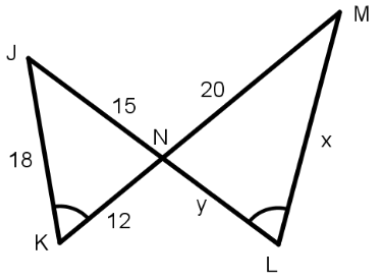


$\angle ABC =$ _____

$\triangle ABC \sim$ _____

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(11) Look at the adjacent figure, and complete:



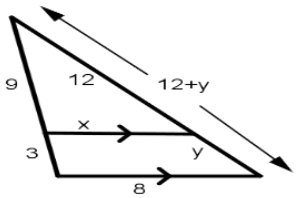
(a) $\triangle JKN \sim \ldots\ldots\ldots$

(b) $\frac{15}{\ldots} = \frac{18}{\ldots}, \frac{15}{\ldots} = \frac{12}{\ldots}$

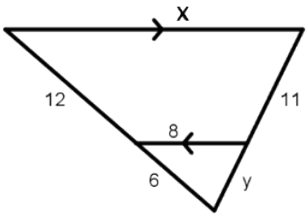
(c) $x = \ldots, y = \ldots$

Find the values of x, y in the following exercises:

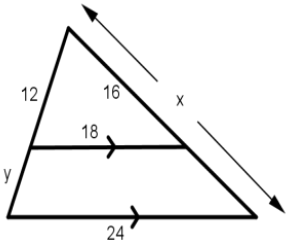
(12)



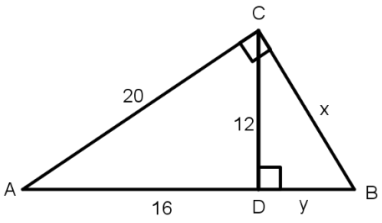
(13)



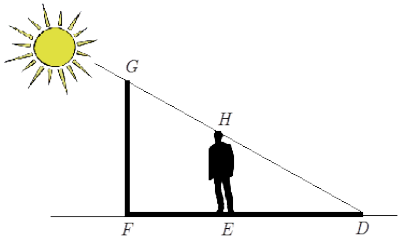
(14)



(15) In the adjacent figure:
(a) Give two triangles that are similar to ABC.
(b) Find the values of x and y.

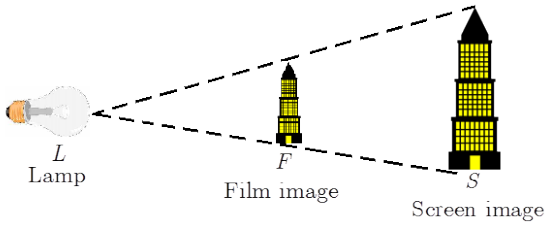


(16) To approximate the height of the basketball stand, one of the players whose height is exactly 2 m stood so that his shadow ended at the end of the stand's shadow. We found that $DE = 1.6$ m and $DF = 4.4$ m.
Find the height of the stand.

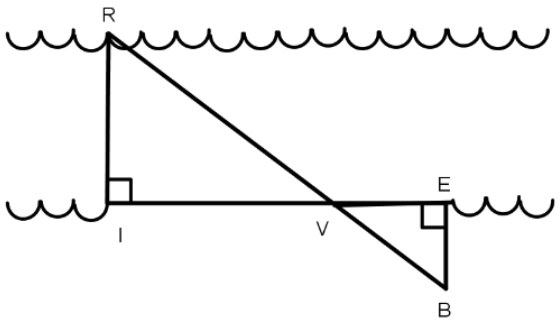


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(17) In the following figure (not drawn to scale), a picture of a building appears on the film as well as the screen (the light source projects the film image on the screen). If $LF = 6$ cm, $LS = 24$ m, and the building height on the screen is 2.2 m, what is the height of the building on the film?



(18) In the following figure, if $IV = 63$ m, $VE = 20$ m and $BE = 15$ m, find the width of the river IR.



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Similarity Postulates (SAS) and (SSS)

Theorem 4:
Two triangles are similar if an angle of one triangle is equal to the corresponding angle of the other, and the ratios of the two pairs of corresponding sides forming these angles are equal. We call this case (SAS) (side-angle-side).

Theorem 5:
Two triangles are similar if the ratios of the three pairs of corresponding sides are equal. We call this similarity (SSS) (side-side-side).

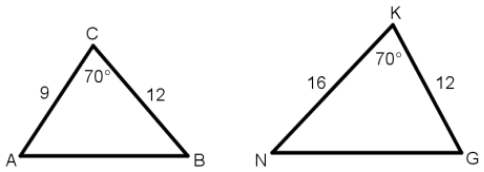
Conclusion:
For any two similar triangles we have:
(1) Corresponding sides are proportional.
(2) Corresponding altitudes are proportional and have the same ratio as the similarity ratio.
(3) Corresponding medians are proportional and have the same ratio as the similarity ratio.
(4) Corresponding angle bisectors are proportional and have the same ratio as the similarity ratio.
(5) The ratio of their perimeters is equal to the similarity ratio.
(6) The ratio of their areas is the square of the similarity ratio.

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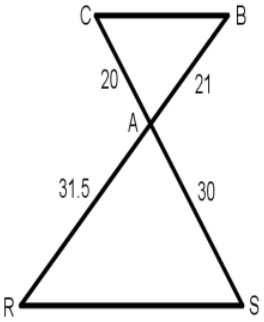
Exercises:

In (19–24), determine the similar triangles and the similarity postulate that you used.

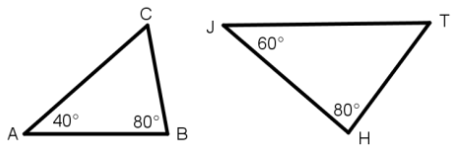
(19)



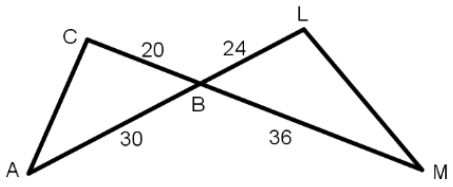
(20)



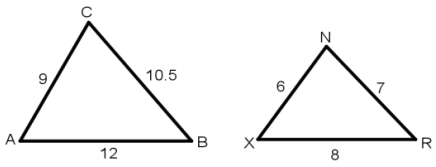
(21)



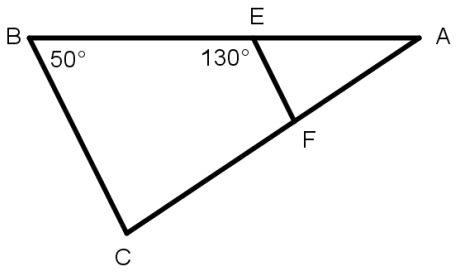
(22)



(23)



(24)



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Proportional Lengths

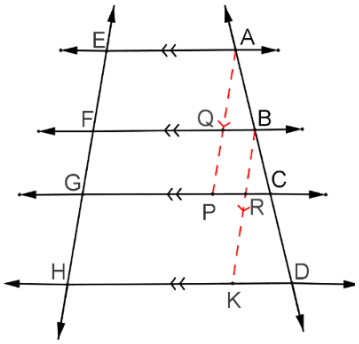
Theorem 6: (Thales Theorem)
If two lines are intersected by several parallel lines, the segments formed on one transversal are proportional to the segments formed on the other transversal.

Proof:

Here $AE \parallel BF \parallel GC \parallel HD$, and these lines are cut by the transversals AD and EH. We need to show that:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}$$

To prove this theorem, we draw AP and BK parallel to EH as shown in the adjacent figure.



Since $AQFE$, $QPGF$, $BRGF$ and $RKHG$ are parallelograms, we get $AQ = EF$, $BR = QP = FG$ and $RK = GH$.

In $\triangle APC$, $BQ \parallel PC$, hence:

$$\frac{AB}{BC} = \frac{AQ}{QP} \Rightarrow \frac{AB}{BC} = \frac{EF}{FG} \Rightarrow \frac{AB}{EF} = \frac{BC}{FG}$$

Similarly, in $\triangle BKD$, $CR \parallel KD$, we obtain:

$$\frac{BC}{CD} = \frac{BR}{RK} \Rightarrow \frac{BC}{CD} = \frac{FG}{GH} \Rightarrow \frac{BC}{FG} = \frac{CD}{GH}$$

Hence:

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH}$$

Theorem 7:
If a line is drawn parallel to one side of a triangle and intersects the other two sides, it divides them into segments proportional in length.

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Theorem 8: (Angle Bisector Theorem)
If the vertex angle of a triangle (or its external angle) is bisected internally or externally, the ratio of the segments of the base equals the ratio of the lengths of the other two sides.

Proof:

In Figure 1, ray AD bisects $\angle CAB$ and intersects BC at D.

In Figure 2, ray AF bisects the external angle $\angle CAE$ and intersects the extension of BC at F.

We need to prove:

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

To prove this, draw $CE \parallel AD$ meeting the extension of BA at E.

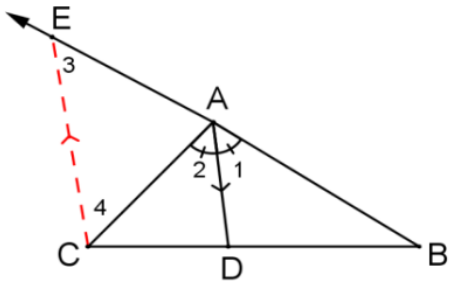


Figure 1

Then $\angle 1 = \angle 3$ (corresponding angles) and $\angle 2 = \angle 4$ (alternate angles).

Since AD bisects $\angle CAB$, $\angle 1 = \angle 2$; hence $\angle 3 = \angle 4$, and therefore $AE = AC$.

In $\triangle BCE$, since $AD \parallel EC$, Theorem 7 gives:

$$\frac{AB}{AE} = \frac{BD}{DC}.$$

Since $AE = AC$, we obtain:

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

In the same way, in Figure 2: draw $CG \parallel AF$ meeting AB at G; then $AG = AC$, and hence:

$$\frac{AB}{AC} = \frac{BF}{FC}.$$

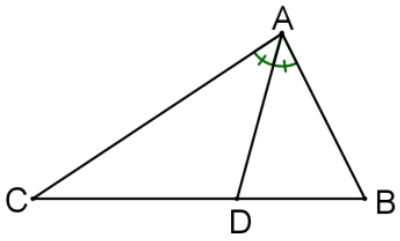
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Notes:

(1) The converse of the theorem holds: if D divides BC such that

$$\frac{AB}{AC} = \frac{BD}{DC}$$

then AD bisects $\angle A$.



(2) AD and AF are the internal and external bisectors of $\angle A$, respectively.

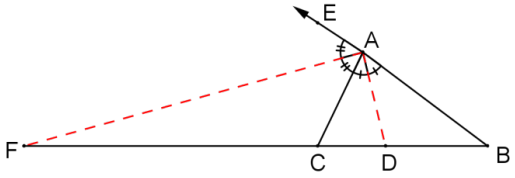


Figure 2

By the angle bisector theorem:

$$\frac{AB}{AC} = \frac{BD}{DC}, \quad \frac{AB}{AC} = \frac{BF}{FC}$$

Thus:

$$\frac{BF}{FC} = \frac{BD}{DC}$$

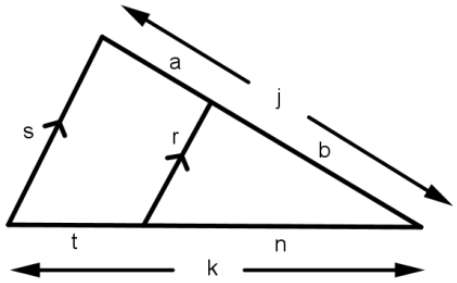
Important Note:

The measure of the angle between the internal and external bisectors of the same angle is $90°$.

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Exercises:

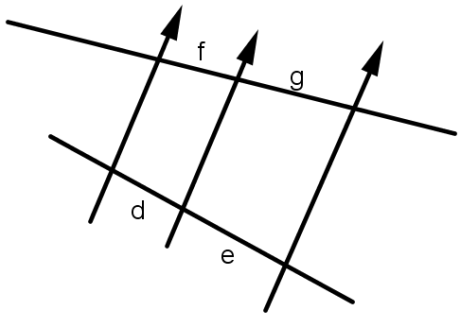
(1) Look at the following figure, and decide which of the relations are correct.



Part	Relation	Answer
(a)	$\frac{r}{s} = \frac{a}{b}$	
(b)	$\frac{t}{k} = \frac{a}{j}$	
(c)	$\frac{j}{a} = \frac{s}{r}$	
(d)	$\frac{r}{s} = \frac{n}{k}$	
(e)	$\frac{a}{b} = \frac{n}{t}$	
(f)	$\frac{b}{j} = \frac{t}{k}$	

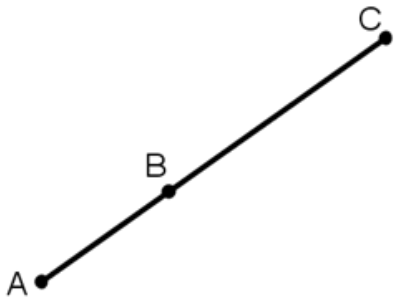
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(2) Look at the following figure, and decide which of the ratios are correct.



Part	Relation	Answer
(a)	$\frac{d}{f} = \frac{g}{e}$	
(b)	$\frac{f}{g} = \frac{e}{d}$	
(c)	$\frac{g}{f} = \frac{e}{d}$	
(d)	$\frac{d}{f} = \frac{e}{g}$	

In (3–6), if $\frac{AB}{BC} = \frac{3}{5}$, complete the other entries of the table.

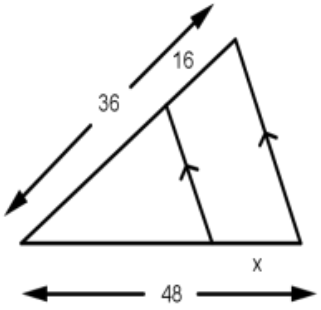


	(3)	(4)	(5)	(6)
AB	6			
BC		25		
AC			56	100

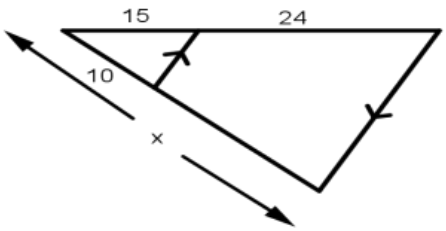
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In (7–14), find the value of x:

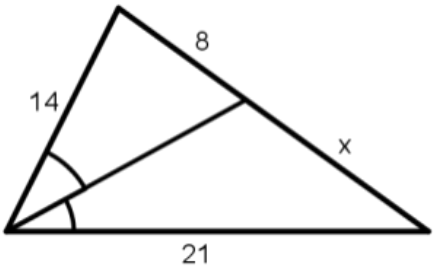
(7)



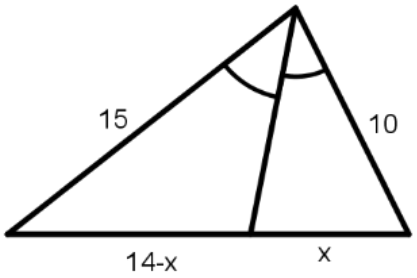
(8)



(9)

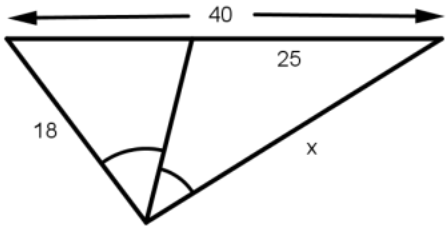


(10)

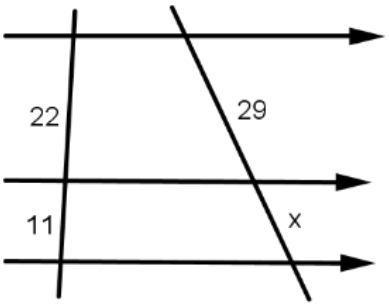


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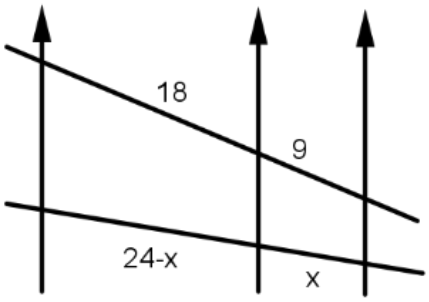
(11)



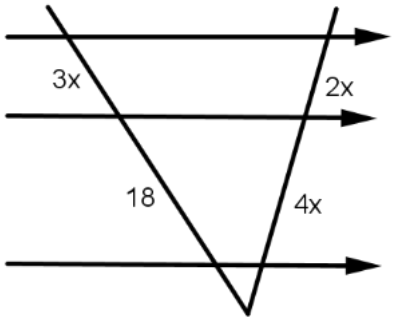
(12)



(13)



(14)



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Third Unit: Number Theory



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Even and Odd Integers

Integers can be divided into two types of numbers: those that are divisible by 2 are called **even** numbers, and the rest of the numbers that are not divisible by 2 are called **odd** numbers. An even integer can be written in the form $2k$, where k is an integer. An odd integer can also be written in the form $2k + 1$, where k is an integer. Any given integer is exclusively either even or odd; it cannot be both.

Other Properties of Even and Odd numbers:

(1) Odd $\neq$ Even.
(2) odd $+$ even $=$ even $+$ odd $=$ odd, odd $-$ even $=$ even $-$ odd $=$ odd.
(3) even $\pm$ even $=$ odd $\pm$ odd $=$ even.
(4) If the product of two integers is even, then at least one of them is even.
(5) The product of two consecutive integers has to be even.
(6) If the sum (or difference) of some integers is odd, then the number of odds in those numbers is odd.
(7) If the sum (or difference) of some integers is even, then the number of odds in those numbers is even.
(8) If the product of some integers is odd, then all of those integers must be odd.
(9) If the product of some integers is even, then at least one of those integers must be even.

Example 1:

Divide the following numbers into even and odd: $102,\ 203,\ 519,\ 3340,\ 70015,\ 87654$. Then find a pattern for even and odd numbers.

Solution:

The number 102 is even since we can write it as $102 = 2(51)$.

203 is odd since $203 = 2(101) + 1$.

519 is odd since $519 = 2(259) + 1$.

3340 is even since $3340 = 2(1670)$.

$70015 = 2(35007) + 1$ so it is an ___ number.

$87654 = 2(43827)$ so it is an ___ number.

If the units digit is $0, 2, 4, 6, 8$ then the number is ___.

If the units digit is $1, 3, 5, 7, 9$ then the number is ___.

Example 2:

If we add two even numbers, then the result would be:

(a) Even (b) Odd

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Solution:

The answer is (a). We know from the properties that the sum of two even numbers is even. Moreover, if we add two digits from $0, 2, 4, 6, 8$, the units digit of the result is still one of $0, 2, 4, 6, 8$.

Can you use the same logic to determine the following:

If we add two odd numbers, then the result would be:

(a) Even (b) Odd

If we add an even and an odd number, then the result would be:

(a) Even (b) Odd

Example 3:

If we pick three random integers a, b, c, then the numbers:

$$\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$$

(a) all of them are non-integers;

(b) at least one of them is an integer;

(c) at least two of them are integers;

(d) all of them are integers.

Solution:

The answer is (b). There are two cases.

Case 1: All of the numbers have the same parity (all odd or all even), which means that sum of any two is even so it is divisible by 2. This leads to $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ being all integers.

Case 2: One of them has a different parity (one odd and two even, or one even and two odd). For example, if a, c are odd while b is even, then $a+b$ and $b+c$ are odd while $a+c$ is even. Thus, only one of $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$ is an integer. By the same logic, every other parity pattern also gives exactly one integer, which gives the answer (b).

Example 4:

The sum of 100 positive integers is 10000, and there are more odd numbers than even numbers among them. Find the largest possible number of even numbers.

Solution:

Let the number of even numbers be x and the number of odd numbers be y. Then

$$x + y = 100, \quad y > x$$

Since the sum 10000 is even, y has to be even (see the properties of even and odd numbers). Thus, y is at least 52, and that means that x is at most 48. It suffices to give an example:

$$\underbrace{1+1+.....+1}_{52 \text{ times}} + \underbrace{2+2+....+2}_{47 \text{ times}} + 9854 = 10000$$

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Thus, the answer is 48.

Example 5:

We have n numbers $x_1, x_2, x_3, \ldots, x_n$, each of which is either 1 or -1. If:

$$x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 = 0$$

What is the most precise statement about n?

(a) even.

(b) odd.

(c) n is a multiple of 4.

(d) cannot say anything about n.

Solution:

The answer is (c). Notice that the sum

$$x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1$$

contains n terms each of which is either 1 or -1. Thus, n has to be even so that the sum is 0. Now we can write $n = 2k$, this means that the numbers $x_1x_2, x_2x_3, ..., x_{n-1}x_n, x_nx_1$ have k of them equal to 1 while the other k are equal to -1. But notice that their product is:

$$(x_1x_2) \cdot (x_2x_3) \cdot \ldots \cdot (x_nx_1) = x_1^2x_2^2 \ldots x_n^2 = 1$$

Thus:

$$(-1)^k(1)^k = 1$$

This means that k should also be even. Therefore, $k = 2m \Rightarrow n = 2k = 4m$ which gives us that n is a multiple of 4.

Note that the converse also holds: for every $n = 4m$ such numbers exist. For example, take

$x_1 = x_4 = x_5 = x_8 = \ldots = x_{4m-3} = x_{4m} = 1$,

$x_2 = x_3 = x_6 = x_7 = \ldots = x_{4m-2} = x_{4m-1} = -1$.

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Exercises:

(1) Find the parity of the number

$$\frac{1221450987543567886333454214938}{2}$$

without finding the value of the number.

(2) We have 50 books that we would like to put in 5 boxes. Can we divide the books into the boxes so that each box has an odd number of books? Explain your answer.

(3) Let a, b be two consecutive integers, $c = ab$ and $N^2 = a^2 + b^2 + c^2$. Determine the parity of N (odd or even) and show your reasoning.

(4) Challenge: In the following pattern $1, 2, 5, 13, 34, 89, \ldots$ in any three consecutive numbers, the sum of the first and third numbers is equal to three times the number in the middle. For example, for $1, 2, 5$ we have $1 + 5 = 3(2)$; for $2, 5, 13$ we have $2 + 13 = 3(5)$, and so on. What is the parity of the 2003^{rd} number?

(5) Let x, y, z be integers, two of them odd and the third even. Show that

$$(x + 1)(y + 2)(z + 3)$$

is even.

(6) We place "$+$" or "$-$" signs between the numbers $1, 2, 3, \ldots, 2018$ in any way we like, then compute the result. Prove that the result is always odd.

(7) (a) We have the equation

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 = 0$$

Can we replace the stars with $+$ and $-$ signs to make the statement correct?
(b) The same question but for the statement:

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 = 0$$

(8) In a chess competition, each player will play 20 games. Wins add 3 points to the total score, draws add one point to the score, and losses subtract one point from the total score. Can a player get a final score of 39? Show your reasoning.

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(9) (a) There are 10 baskets on a circle. Can we put oranges in each basket so that the difference in the number of oranges between each consecutive baskets is 1?
(b) What if there were 9 baskets? Show your reasoning.

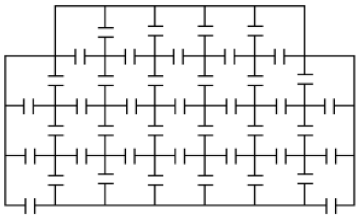
(10) A worm moves on a straight line; it can jump 6 or 8 cm in either direction (left or right). Can the worm reach a point that is:

(a) 1.5 cm (b) 7 cm (c) 4 cm

away from its original position?

(11) Challenge: Can we arrange the 10 numbers $1, 1, 2, 2, 3, 3, 4, 4, 5, 5$ in a line so that there is one number between $1, 1$, two numbers between $2, 2$, three numbers between $3, 3$, four numbers between $4, 4$, and five numbers between $5, 5$? Justify your answer.

(12) Challenge: Can we visit the 26 rooms in the diagram so that we enter each room exactly once? Justify your answer.



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Divisibility

Definition

We say a number a divides b if and only if b can be written as $a \times k$ where k is an integer. Formally we write it as $a \mid b$ and read it as "a divides b" or "b is divisible by a".

For example:

$2 \mid 10$ because $10 = 2 \times 5$

$3 \mid -27$ because $-27 = 3 \times (-9)$

$6 \mid 0$ because $0 = 6 \times 0$

$6 \nmid 10$ because there is no integer k with $6k = 10$

Properties of divisibility:

$a \mid b$ if and only if $b = ak$ for some integer k.
If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.
If $a \mid b$ and $a \mid c$, then $a \mid (b - c)$.
If $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m, n.
If $a \mid b$ and $b \mid c$, then $a \mid c$.
If $a \mid b$, then $a \mid bc$ for every integer c.
If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Example 1:

Find all positive integer values of n that satisfy $n - 1 \mid 3n + 1$.

Solution:

$n = 2, 3, 5$

Note that $n-1 \mid 3(n-1) = 3n-3$ and by the subtraction property we get $n-1 \mid 3n+1-(3n-3) = 4$ and divisors of 4 are $1, 2, 4$ (we reject the negative since n is positive) hence $n - 1 = 1, 2,$ or 4, so $n = 2, 3,$ or 5.

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Exercises:

(1) Prove the properties of divisibility using the definition.

(2) Suppose that: $11 \mid (a+3)$ and $11 \mid (b-4)$. What is the remainder of $b-a$ when divided by 11?

(3) We know that: $n \mid 100$ and $n \mid 36$. What are all possible positive values of n?

(4) Suppose that $n \mid 2a+3b$ and $n \mid 5a+7b$. Show that $n \mid a$ and $n \mid b$.

(5) If $a \mid b$ and $a+b=100$, find all possible positive values of a and b.

(6) Find all positive integers n such that $(n^2+5n+11)/(n+2)$ is an integer.

(7) Show that $6 \mid n^3-n$ for any positive integer n.

(8) Challenge: Find all ordered pairs of positive integers (a,b) such that $a \mid b+1$ and $b \mid a+1$.

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Prime and Composite Numbers

Revision

- Among all the positive integers, the number 1 is the only one with exactly one positive divisor, which is itself.
- All positive integers larger than 1 have at least two positive divisors.
- If a positive integer has only two positive divisors (itself and 1), then we call it a **prime** number.
- All positive integers larger than 1 that are not prime are called **composite** numbers.

From the previous definitions, we conclude the following:
The number 1 is neither a prime nor a composite number.
There is only one even prime number, 2, and it is the smallest prime.
The smallest composite number is 4.

Every integer $N > 1$ can be factorized into prime factors:

$$N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k},$$

where $p_1, p_2, p_3, \ldots, p_k$ are different primes and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are non-negative integers. Then the number of positive divisors of N is:

$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \cdots (\alpha_k + 1)$$

Example 1:

If p and $p^3 + 5$ are primes, then $p^5 + 7$ is:
(a) prime. (b) composite. (c) prime or composite. (d) neither prime nor composite.

Solution:

(b) composite number.

Notice that if p is odd then $p^3 + 5$ is an even number larger than 2 and that makes a composite number. Thus $p = 2$, which makes $p^5 + 7 = 2^5 + 7 = 39 = 13 \times 3$ which is a composite number.

Example 2:

Given three primes p, q, r such that $p + q = r$ and $p < q$, find the value of p.

Solution:

Notice that r cannot be even: the only even prime is 2, but $r = p + q \geq 2 + 3 = 5$. Then, r is odd, which makes p, q one even and one odd number. However, we know that the only even prime is 2. Moreover, $p < q$, which leads to $p = 2$.

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Exercises:

(1) Given that x and y are primes, find the number of ordered pairs (x, y) with $x + y = 75$.

(a) 1 (b) 2 (c) 3 (d) 4

(2) We call a two-digit prime $\overline{ab}$ naughty if $\overline{ba}$ is also prime. How many naughty primes are there?

(3) The distinct primes a, b, c satisfy $a \cdot b^b \cdot c + a = 2000$. Find all possible values of $a + b + c$.

(4) Given that p, q and $p - q$ are primes and $p + q$ is even, find the value of:

$$\left(1 + \frac{1}{2}\right)^p \left(1 - \frac{1}{3}\right)^q$$

(5) Challenge: Let n be a positive integer such that $n + 3$ and $n + 7$ are both prime. Find the remainder when n is divided by 3.

(6) Challenge: If p is a prime not less than 5 and $2p+1$ is also prime, prove that $4p+1$ is composite.

(7) We have the equation $56a = 65b$ with a, b positive integers. Prove that $a + b$ is a composite number.

(8) Given that p, q, r are prime numbers satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1661}{1986}$$

find the value of $p + q + r$.

(9) Let p, q be primes with $p = m + n$ and $q = mn$, where m, n are positive integers. Find the value of

$$\frac{p^p + q^q}{m^n + n^m}$$

(10) Given that p, q are prime numbers satisfying $5p + 7q = 129$, find all possible values of $p + q$.

(11) Challenge: If $p, p + 2, p + 6, p + 8, p + 14$ are all prime, find the value of p.

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(12) Let p, q be primes, and let the integer n satisfy

$$\{n-1, 3n-19, 38-5n, 7n-45\} = \{p, 2p, q, 2q\}$$

(not necessarily in this order). Find n.

(13) Challenge: Prove that there are infinitely many positive integers n for which n^2+n+41 is a composite number.

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Fourth Unit: Combinatorics



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Counting Principles

Before going deeper into new counting methods, it is essential to review the concepts we covered earlier, because they are the foundation of all the remaining counting methods.

Review: The Two Basic Counting Principles

Multiplication Principle: If event A can occur in m different ways and event B can occur in n different ways, and the two events are independent, then the two events can occur together in $m \cdot n$ ways.

Addition Principle: If event A can occur in m different ways and event B can occur in n different ways, and the two events are mutually exclusive, then event A or event B can occur in $m + n$ different ways.

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Exercises:

(1) A restaurant offers meat sandwiches in three sizes (small, medium, or large), with cheese, with tomato, or without either. Faisal wants to order a cheese sandwich. In how many ways can he do that?

(2) A shoe store has 6 types of shoes, and each type is available in 7 colors. How many different shoes are available in this store?

(3) Salman, Osama, and Khalid are running a race. In how many ways can the race end if it is possible for 2 or more of them to tie?

(4) The National Bank asks its new customers to choose a PIN made up of 4 different digits from the numbers 1 to 5, with the condition that the absolute value of the difference between each digit and the one after it is at least 2. In how many ways can a customer choose the PIN?

(5) In how many ways can 3 different mathematics books, 4 different chemistry books, and 5 different physics books be arranged on a bookshelf, provided that the books of each subject are next to one another?

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Counting Numbers and Digit Strings

This is a common type of problem that can be solved using the counting principles, in which we are asked to find how many numbers satisfy a certain condition. To clarify the upcoming problems, you may need the following definitions:

A natural number with n digits: We say that a natural number consists of n places (digits) if the digit in its highest place (the first digit from the left and the n-th from the right) is nonzero.

For example: the number 2345 consists of four digits, but we do not consider the number 034 to consist of three digits, because the zero in the highest place does not count; the number is written in the form 34, that is, it consists of only two digits.

A string of n digits:
We say that a string consists of n places regardless of whether the digit in its highest place is zero or not.

For example: each of 01234, 00034, and 12302 is a string consisting of five places.

Palindromic numbers:
We say that a natural number is palindromic if reading its digits from right to left is identical to reading them from left to right.

For example: the numbers 1234321, 919, and 6 are palindromic, while the number 2342 is not palindromic.

The two counting principles will help us count the required numbers or strings in a simple yet deep way. Let us begin the journey of creativity!

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Exercises:

(6) How many positive integers consist of four different digits?

(7) How many positive odd integers consist of five digits?

(8) How many two-digit numbers can be formed from the digits of the set $\{0, 3, 4, 5, 6, 7, 8, 9\}$, provided that the sum of their digits is odd?

(9) How many numbers between 0 and 1000 contain the digit 5 exactly once?

(10) How many numbers that are divisible by 3 and have 3 different digits can be formed from the digits of the set $\{1, 3, 7, 8, 9\}$?

(11) How many strings of 10 digits, consisting of ones and zeros only, contain exactly five adjacent zeros?

(12) How many five-digit numbers satisfy the condition that the sum of their first digit and their fifth digit equals 5?

(13) How many six-digit numbers have all of their digits of the same type (all even or all odd)?

(14) How many positive integers with 4 digits have exactly one digit "1" and exactly one digit "3"?

(15) How many numbers consisting of 3 digits have at least one even digit?

(16) How many four-digit numbers are there in which no two even digits are adjacent?

(17) How many positive integers less than 1000 do not contain the digit 7?

(18) How many positive integers consisting of five different digits satisfy the condition that the positive difference between the first digit and the last digit equals 2?

(19) When writing the numbers from 1 to 100, how many times will we write the digit 6?

(20) How many palindromic numbers are there with: (a) 6 digits formed from the digits $\{1, 2, 3, 4, 5, 6, 7, 8\}$? (b) 7 digits formed from the same digits as above? (c) 5 digits formed from the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$?

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Counting Words

The number of English words consisting of n letters can be calculated directly using the multiplication principle. In counting problems, the resulting words do not necessarily have to have a meaning.

Example:

The number of words consisting of 4 letters equals:

First letter	Second letter	Third letter	Fourth letter	**Number of words**
Any letter	Any letter	Any letter	Any letter	
26 ways $\times$	26 ways $\times$	26 ways $\times$	26 ways $=$	26^4

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Exercises:

(21) How many words consist of 5 different letters?

(22) In how many ways can the letters of the word PRODUCT be arranged?

(23) How many words consist of 6 letters, provided that the first, third, and sixth letters are different?

(24) How many words made up of 3 letters contain the letter M exactly once?

(25) How many words made up of 3 letters contain the letter M at least once, if repeating any other letter is not allowed?

(26) How many words made up of 3 letters contain the letter M at least once, if repeating another letter is allowed (another letter may appear twice)?

(27) The Saudi team for the Cybersecurity Olympiad invented its own language consisting of only four letters. In their language, the longest word contains at most 6 letters. How many words does this language have?

(28) We define a good word as a word made up only of the letters A, B, and C (it is possible for one of these letters not to appear in the word) such that the letter B never immediately follows the letter A, the letter C never immediately follows the letter B, and the letter A never immediately follows the letter C. How many good words consist of 7 letters?

(29) How many palindromic words consist of seven letters and do not begin with a vowel? The vowels are (A, E, I, O, U).

(30) In how many ways can all the English letters be written in a row, provided that there are exactly five letters between the two letters x and y?

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Permutations

We begin with an example to illustrate the idea behind permutations.

Example:

In how many ways can the friends Ahmed, Mohammed, Younes, Hamza, and Hashem stand in a single row to take a group photo? And in how many ways can 3 of them stand in a row to take the photo?

Solution:

When arranging all five friends in a row: there are 5 choices for the first position; after that, there are 4 ways to choose the second, 3 ways for the one in the middle, 2 ways for the one before last, and 1 choice will remain for the last one among them.

The total number of ways is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

When arranging only three of them: by the same method, the number of ways will be $5 \times 4 \times 3 = 60$.

By noticing the pattern in the solution of this question and of a number of other questions, we can deduce a rule that works whenever the goal is to count the ways of choosing a number of elements from some set (a number of people in the previous example) and then arranging them.

Permutations: If we have a set containing n objects and we want to choose k of them, taking order into account and without repetition, then the number of choices is:

$$^{n}P_{k} = n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$$

It is read "n permute k".

Now that we have learned the concept of permutations, let us return to the previous example:

Example:

In how many ways can the friends Ahmed, Mohammed, Younes, Hamza, and Hashem stand in a single row to take a group photo? In how many ways can 3 of them stand in a row to take the photo? And in how many ways can at least 3 of them stand in a row to take the photo?

Solution:

When arranging all five friends in a row: the number of ways is $^{5}P_{5} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

When arranging only three of them: the number of ways is $^{5}P_{3} = 60$.

When arranging at least three of them: the number of ways is $^{5}P_{3} + ^{5}P_{4} + ^{5}P_{5} = 60 + 120 + 120 = 300$ ways.

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Note:

We can write $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ as $5!$, which is read "5 factorial".

Factorial: The factorial of n is the product of the positive integers less than or equal to n, and it is denoted by the exclamation mark $(!)$:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Notice that:

$$^{n}P_{n} = n \times (n-1) \times (n-2) \times \ldots \times 1$$

This represents choosing all n of the objects, taking order into account.
Permutations can also be written in the form:

$$^{n}P_{k} = \frac{n!}{(n-k)!}$$

Calculate $3!$, $3! \times 4$, and $4!$. What do you notice?

Corollary: For any positive integer n, we have $n! = (n-1)! \cdot n$.

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Exercises:

(31) A football stadium has fifteen gates. In how many ways can you enter through one gate and exit through another gate?

(32) There are 7 lamps in a meeting hall. To keep the lighting suitable for the hall, at least one of the lamps must be on. In how many ways can this be done?

(33) We have the set $\{1, 2, 3, \ldots, 8\}$.
(a) How many subsets does it have?
(b) How many of its subsets contain no odd number?
(c) How many of its subsets contain exactly one odd number?

(34) We have a square board divided into identical squares, with board dimensions 2026×2026, and we want to shade squares of the board so that exactly one square is shaded in each row and in each column. (Assume that rotation and reflection give different configurations.)
(a) In how many ways can this be done?
(b) If we require that the corner squares are not shaded, in how many ways can this be done?

(35) How many four-digit numbers have only two identical digits and have a digit less than 3 in the thousands place?

(36) From a set containing 9 elements, how many non-empty subsets contain an even number of elements?

(37) How many numbers consisting of 5 different digits are there such that the first and last digits are even numbers?

(38) In a football league, the organizers want to choose 3 teams to be honored at the annual ceremony, so that the chosen teams get different places: first, second, and third. Each winning team chooses one of its players to receive the award. If the league has 12 teams and each team has 25 players, in how many ways can the three teams be chosen, along with specifying the players?

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Answers



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A note to the student

We believe that learning mathematics is about thinking hard about a problem and getting stuck — and pushing through. That is how real problem-solving ability is built. This is why this section gives only the final answers, not the full solutions: try every problem on your own first. If you want to check the full solutions, watch the NSMO videos.

Algebra Answers

Percentages:

(1) 30

(2) 30%

(3) 18 cookies

(4) 162°

(5) $a : c = 17 : 21$

(6) 80%

(7) 3125 fish

(8) $4t : 2u = 1 : 2$

(9) $k = \frac{1}{4}$

(10) $k = 18$

(11) $a = 5, b = 2, c = 1$

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One-Variable Linear Equations:

(1) $x = -\frac{74}{167}$

(2) $x = a + b + c$

(3) No real solution.

(4) If $a \neq 1$: $x = \frac{5-20b+8ab}{20(a-1)}$; if $a = 1$, $b = \frac{5}{12}$: every real x is a solution; if $a = 1$, $b \neq \frac{5}{12}$: no solution.

(5) $a = \frac{5}{3}$, $b = \frac{26}{9}$

(6) $a = 2$

(7) 6 values ($k = 1, 2, 3, 4, 6, 12$)

(8) Only one value: $x = 2$

(9) 3

(10) $x = \frac{n^2}{n-m}$

Multi-Variable Linear Equations:

(1) Infinitely many solutions.

(2) $(a)\ x = 10,\ y = 3$; $(b)\ x = 4,\ y = 1$; $(c)\ x = 8,\ y = 6$; $(d)\ x = 5,\ y = 4$

(3) 25 and 17

(4) $3{:}09$

(5) $x = \frac{59}{4}$, $y = -\frac{11}{4}$

(6) $x = 15$, $y = 5$

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(7) $x = 2, y = -3$

(8) $x = \frac{30}{41}, y = \frac{45}{41}, z = \frac{75}{41}$

(9) $x = 1, y = 2, z = 3, u = 4$

(10) $x = \frac{2a+b-c}{14}, y = \frac{2b+c-a}{14}, z = \frac{2c+a-b}{14}$

(11) $4a - c = 9$

(12) $(a, b, c) = (2, 3, 4)$

(13) (a) $k \neq -2$; (b) no value of k; (c) $k = -2$

(14) $z - y = 2021$

(15) $x = 7, y = 5, z = -3$

(16) $x = 0, y = 6, z = 7, u = 3, v = -1$

Challenge Problems:

(1) $a = \frac{1}{10}, b = \frac{1}{4}$

(2) $x = 5, y = 3, z = 1$

(3) E.g. $\frac{1}{2}$ and $\frac{1}{3}$ (in general $\frac{1}{a}$ and $\frac{1}{a+1}$)

(4) $3 : 7$

(5) 23 children

(6) $c = 1$

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(7) 553451234512345

(8) Construction problem — see the NSMO videos.

(9) Nour is a girl (she is 5 years old).

(10) 25 liars

(11) No; at most 16 squares can be colored.

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Geometry Answers

Revision Exercises:

(1) $4n + m = 180°$

(2) $\angle A = 23°$

(3) $\angle EAD = 141°$

(4) $x = a + c + d - b$

(5) $\angle DBF = 36°$ (since $\angle BDF = 90°$)

(6) $(a)\ \angle BAC = 2b - 2a$, $\angle BDC = b - a$; $(b)\ a = 39°$, $b = 58.5°$

(7) $y = 2x$

(8) $\angle EDC = 20°$, $\angle BDC = 90°$

(9) $\angle ACE = \frac{180° - 3x}{2}$

(10) $a = 42°$, $b = 84°$, $c = 63°$, $d = 14°$

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Similarity:

(1–8)

Exercise	Polygons	Always similar	Can be similar	Cannot be similar
(1)	**Equilateral triangles**	✓		
(2)	**Isosceles triangles**		✓	
(3)	**Squares**	✓		
(4)	**Rhombuses**		✓	
(5)	**Right triangles**		✓	
(6)	**Scalene triangles**		✓	
(7)	**Rectangles**		✓	
(8)	**Right triangle and acute triangle**			✓

(9–16)

Exercise	Required	Answer
(9)	Similarity ratio of $ABCD$ to $A_1B_1C_1D_1$	$\frac{3}{4}$
(10)	Type of quadrilateral $A_1B_1C_1D_1$	Trapezoid
(11)	Measure of $\angle D_1$	45°
(12)	Measure of $\angle C_1$	$180^\circ - 45^\circ = 135^\circ$
(13)	Measure of C_1B_1	$\frac{12}{9} = \frac{C_1B_1}{6} \Rightarrow C_1B_1 = 8$
(14)	Measure of AD	$\frac{12}{9} = \frac{20}{AD} \Rightarrow AD = 15$
(15)	Measure of C_1D_1	$\frac{12}{9} = \frac{C_1D_1}{3k} \Rightarrow C_1D_1 = 4k$
(16)	Ratio of the perimeters of the quadrilaterals	$\frac{3}{4}$

(17) $x = \frac{65}{4}, y = \frac{64}{5}, z = \frac{56}{5}$

(18) $x = \frac{126}{5}, y = \frac{108}{5}, z = \frac{162}{5}$

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Triangle Similarity:

$(1–9)$

Exercise	Answer
(1)	Similar
(2)	Not similar
(3)	Similar
(4)	Similar
(5)	Similar
(6)	Inconclusive
(7)	Similar
(8)	Inconclusive
(9)	Similar

(10) $\angle ABC = 40°$; $\triangle ABC \sim \triangle GEF$

(11) (a) $\triangle JKN \sim \triangle MLN$; (b) $\frac{15}{20} = \frac{18}{x}$, $\frac{15}{20} = \frac{12}{y}$; (c) $x = 24$, $y = 16$

(12) $x = 6$, $y = 4$

(13) $x = 24$, $y = \frac{11}{2}$

(14) $x = \frac{64}{3}$, $y = 4$

(15) (a) $\triangle ACD$ and $\triangle CBD$ (i.e. $\triangle ABC \sim \triangle ACD \sim \triangle CBD$); (b) $x = 15$, $y = 9$

(16) The height of the stand is $GF = 5.5$ m

(17) The building's height on the film is 0.55 cm

(18) $IR = 47.25$ m

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$(19–24)$

Exercise	Similar Triangles	Theorem
(19)	$\triangle ABC \sim \triangle GNK$	SAS
(20)	$\triangle ABC \sim \triangle ARS$	SAS
(21)	$\triangle ABC \sim \triangle THJ$	AA
(22)	$\triangle ABC \sim \triangle MBL$	SAS
(23)	$\triangle ABC \sim \triangle XRN$	SSS
(24)	$\triangle ABC \sim \triangle AEF$	AA

Proportional Lengths:

(1) True: (b), (d); false: (a), (c), (e), (f)

(2) True: (c), (d); false: (a), (b)

$(3–6)$

	(3)	(4)	(5)	(6)
AB	6	15	21	37.5
BC	10	25	35	62.5
AC	16	40	56	100

(7) $x = \frac{64}{3}$

(8) $x = 26$

(9) $x = 12$

(10) $x = \frac{28}{5}$

(11) $x = 30$

(12) $x = 14.5$

(13) $x = 8$

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(14) $x = 3$

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Number Theory Answers

Even and Odd Integers:

(1) The number is odd.

(2) No; the sum of 5 odd numbers is odd while 50 is even.

(3) N is odd.

(4) The 2003^{rd} term is even.

(5) Proof problem — see the NSMO videos.

(6) Proof problem — see the NSMO videos.

(7) (a) Yes, e.g. $1+2-3+4+5+6-7-8=0$; (b) No (the sum 45 is odd).

(8) No; the total score is always even while 39 is odd.

(9) (a) Yes, e.g. $1, 2, 1, 2, \ldots, 1, 2$; (b) No, it is impossible.

(10) (a) No; (b) No; (c) Yes (e.g. 6 cm twice to the right, then 8 cm to the left).

(11) No, such an arrangement is impossible.

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(12) Impossible.

Divisibility:

(1) Proof problem — see the NSMO videos.

---

(2) The remainder is 7.

---

(3) $n = 1, 2, 4$.

---

(4) Proof problem — see the NSMO videos.

---

(5) $(a, b) = (1, 99), (2, 98), (4, 96), (5, 95), (10, 90), (20, 80), (25, 75), (50, 50)$.

---

(6) $n = 3$.

---

(7) Proof problem — see the NSMO videos.

---

(8) $(a, b) = (1, 1), (1, 2), (2, 1), (2, 3), (3, 2)$.

Prime and Composite Numbers:

(1) Answer (b): two solutions, $(2, 73), (73, 2)$.

---

(2) 9 numbers: $11, 13, 17, 31, 37, 71, 73, 79, 97$.

---

(3) $a + b + c = 42$ (the only value).

---

(4) $\frac{9}{4}$.

---

(5) The remainder is 1.

---

(6) Proof problem — see the NSMO videos.

---

(7) Proof problem — see the NSMO videos.

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(8) $p + q + r = 336$.

(9) $\frac{31}{3}$.

(10) $p + q \in \{19, 25\}$.

(11) $p = 5$.

(12) $n = 7$.

(13) Proof problem — see the NSMO videos.

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Combinatorics Answers

Counting Principles:

(1) 3 ways.

(2) 42 different shoes.

(3) 13 ways.

(4) 20 ways.

(5) $3! \times 3! \times 4! \times 5! = 103680$ ways.

Counting Numbers and Digit Strings:

(6) 4536 numbers.

(7) 45000 numbers.

(8) 28 numbers.

(9) 243 numbers.

(10) 24 numbers.

(11) 64 strings.

(12) 5000 numbers.

(13) 28125 numbers.

(14) 720 numbers.

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(15) 775 numbers.

(16) 4625 numbers.

(17) 728 numbers.

(18) 5040 numbers.

(19) 20 times.

(20) $(a)\ 8^3 = 512$; $(b)\ 8^4 = 4096$; $(c)\ 648$.

Counting Words:

(21) $26 \times 25 \times 24 \times 23 \times 22 = 7893600$ words.

(22) $7! = 5040$ ways.

(23) 274185600 words.

(24) 1875 words.

(25) 1876 words.

(26) 1951 words.

(27) 5460 words.

(28) $3 \times 2^6 = 192$ good words.

(29) $21 \times 26^3 = 369096$ words.

(30) $40 \times 24!$ ways.

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Permutations:

(31) $15 \times 14 = 210$ ways.

---

(32) $2^7 - 1 = 127$ ways.

---

(33) (a) $2^8 = 256$ subsets; (b) 16 subsets; (c) 64 subsets.

---

(34) (a) $2026!$ ways; (b) $2024 \times 2023 \times 2024!$ ways.

---

(35) 864 numbers.

---

(36) 255 subsets.

---

(37) 5376 numbers.

---

(38) 20625000 ways.

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